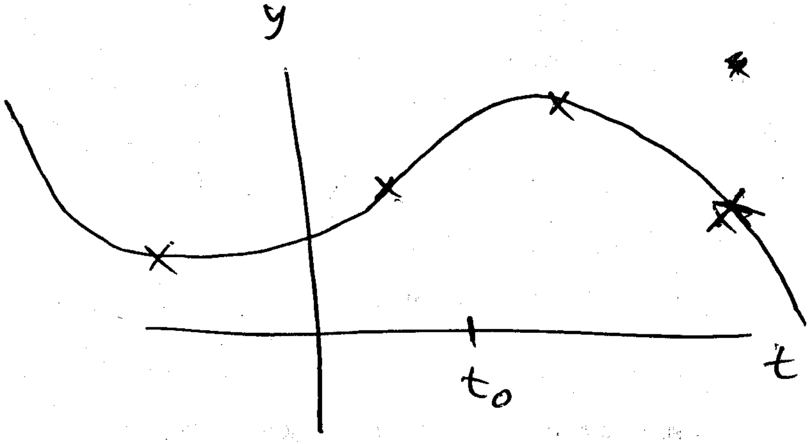
6-11-2013

Interpolation with polynomials.

<u>Idea:</u> You have a few data points



Want to estimate what happens in between.

Find a curve that passes thru the points.

<u>Q:</u> what curve?

We are looking at polynomials.

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \quad (\text{degree } n)$$

<u>Q:</u> How do we find it?

<u>Fact:</u> If we have 4 data points, what degree poly do we look for? 3 (i.e. a cubic poly $y = \underline{a_0} + \underline{a_1} x + \underline{a_2} x^2 + \underline{a_3} x^3$)

If we have n data point, looking for a degree n-1 poly.

<u>Fact:</u> It is possible no solution exists. When 2 different points have same x-value.

example: Find a polynomial interpolating the points (1,3) (2,6) (3,11) (4,12)

Fix the degree: 3 $\quad y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 = 3$$
$$a_0 + a_1(2) + a_2(2)^2 + a_3(2)^3 = 6$$
$$a_0 + a_1(3) + a_2(3)^2 + a_3(3)^3 = 11$$
$$a_0 + a_1(4) + a_2(4)^2 + a_3(4)^3 = 12$$

$$\begin{bmatrix} 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 11 \\ 12 \end{bmatrix}$$

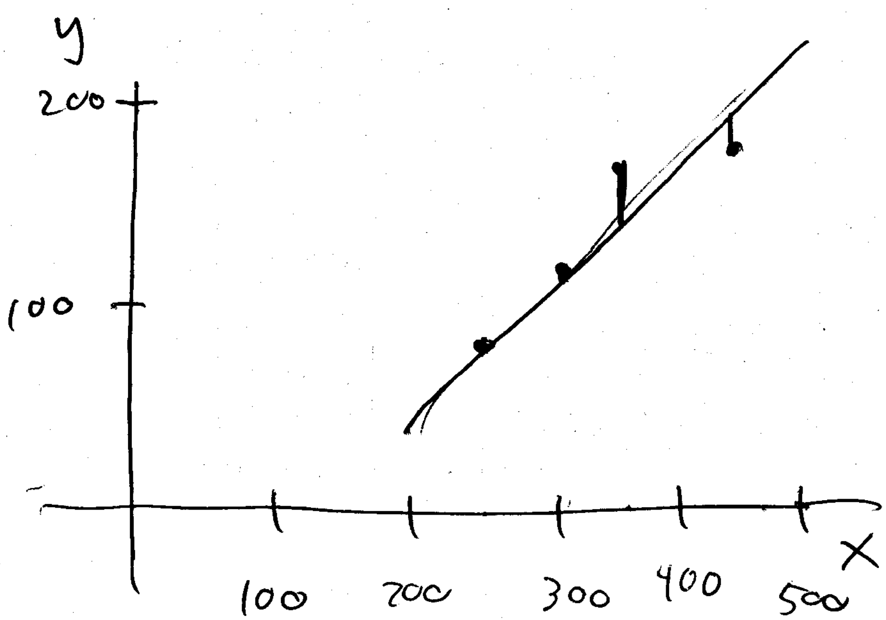
Solve for a_0, a_1, a_2, a_3.

$a_0 = 8$, $a_1 = -11$, $a_2 = 7$, $a_3 = -1$

$\therefore$ $y = 8 - 11x + 7x^2 - x^3$ interpolates the data points.

② Least squares.

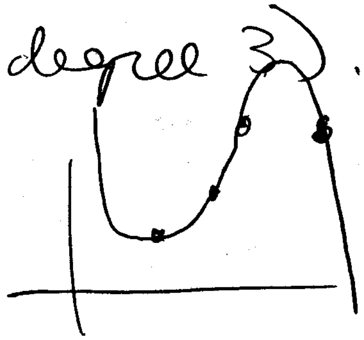
eg #11, p.42 (book)



x = consumption
y = death rate

Want to fit a good model to this data.

Idea: Interpolate with a polynomial.
(degree 3). Bad idea: too much wiggling



Idea: Pick a line that approximates the data well.

Least squares method: minimize the square of vertical distance.

4 points: $(250, 95)$ $(300, 120)$ $(350, 165)$ $(470, 170)$ line: $y = a_0 + a_1 x$

Idea:

$$a_0 + a_1(250) = 95$$
$$a_0 + a_1(300) = 120$$
$$a_0 + a_1(350) = 165$$
$$a_0 + a_1(470) = 170$$

$$\begin{bmatrix} 1 & 250 \\ 1 & 300 \\ 1 & 350 \\ 1 & 470 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 95 \\ 120 \\ 165 \\ 170 \end{bmatrix}$$

$$A \quad \underline{X} = b$$

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 250 & 300 & 350 & 470 \end{bmatrix}$$ "A transpose"

Solve $A^T A \underline{X} = A^T b$.

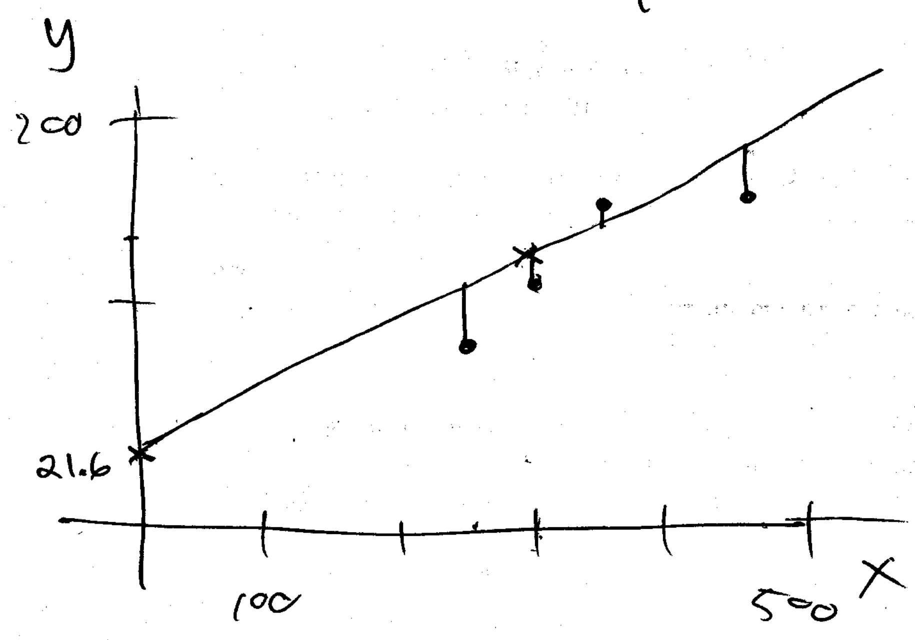
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 250 & 300 & 350 & 470 \end{bmatrix} \begin{bmatrix} 1 & 250 \\ 1 & 300 \\ 1 & 350 \\ 1 & 470 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 250 & 300 & 350 & 470 \end{bmatrix} \begin{bmatrix} 95 \\ 120 \\ 165 \\ 170 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1370 \\ 1370 & 495900 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 550 \\ 197400 \end{bmatrix}$$

Solve the system.

$y = 21.6 + .34x$

$a_0 = 21.6 \quad a_1 = .34$



Predict: if $x = 500$ $y = 21.6 + .34(500)$

≈ 192

eg: Find least squares line for data:

(1, 6) (4, 5) (6, 14). $y = a_0 + a_1 x$

$$a_0 + a_1(1) = 6$$

$$a_0 + a_1(4) = 5$$

$$a_0 + a_1(6) = 14$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 14 \end{bmatrix}$$

$$A \underline{X} = b$$

$A^T A \underline{X} = A^T b$ ← Normal Equations.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 11 \\ 11 & 53 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 25 \\ 110 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 11 & 25 \\ 11 & 53 & 110 \end{bmatrix} \rightarrow \begin{array}{c} -11 \\ \end{array} \begin{bmatrix} 1 & \frac{11}{3} & \frac{25}{3} \\ 11 & 53 & 110 \end{bmatrix}$$

$$\begin{array}{c} \\ \frac{3}{38} \end{array} \begin{bmatrix} 1 & \frac{11}{3} & \frac{25}{3} \\ 0 & \frac{38}{3} & \frac{55}{3} \end{bmatrix} \rightarrow \begin{array}{c} \\ -\frac{11}{3} \end{array} \begin{bmatrix} 1 & \frac{11}{3} & \frac{25}{3} \\ 0 & 1 & \frac{55}{38} \end{bmatrix}$$

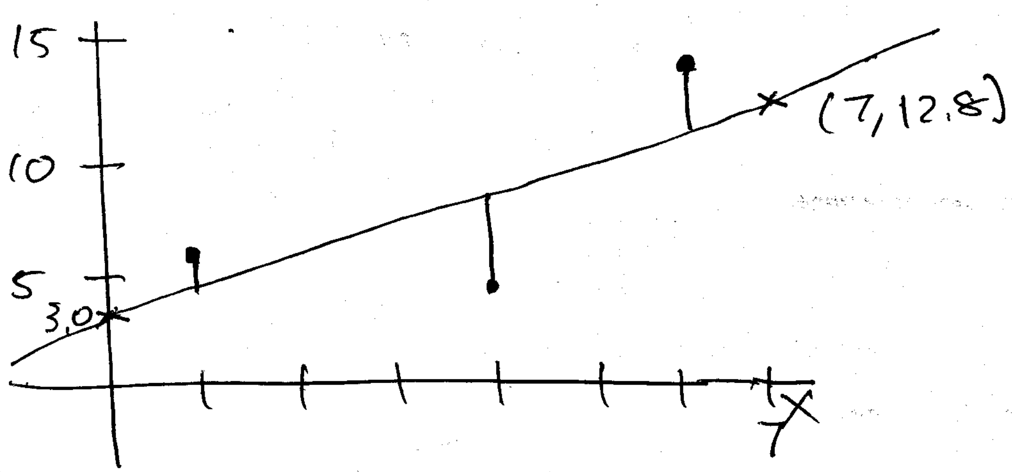
$$-\frac{121}{3} + \frac{159}{3} = \frac{38}{3}$$

$$-\frac{275}{3} + \frac{330}{3} = \frac{55}{3}$$

$$\begin{bmatrix} 1 & 0 & 3.0 \\ 0 & 1 & \frac{55}{38} \approx 1.4 \end{bmatrix}$$

$$-\frac{11}{3} \cdot \frac{55}{38} + \frac{25}{3} \approx 3.0$$

$$y \cong 3.0 + 1.4x$$



eg $(1,3)\ (2,6)\ (3,12)$ $\quad y = a_0 + a_1 x$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 12 \end{bmatrix}$$

$$A^T \quad A \quad X = A^T b$$

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 21 \\ 51 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{bmatrix}$$

$(3)(14) - (6)(6)$

$42 - 36 = 6$

$$\begin{bmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 21 \\ 51 \end{bmatrix} = \begin{bmatrix} -2 \\ 9/2 \end{bmatrix} \quad //$$

$$\boxed{\begin{aligned} y = -2 + \tfrac{9}{2}x &= -2 + 4.5x \\ &= 4.5x - 2 \end{aligned}}$$

eg (1,8) (2,6) (3,4) (4,3) $y = a_0 + a_1 x$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 21 \\ 44 \end{bmatrix}$$

$$A^T A \ X = A^T b$$

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eg MyMathLab Hw 1.5 (6)

$$\begin{bmatrix} 1 & 0 \\ 1 & 5 \\ 1 & 10 \\ 1 & 15 \\ 1 & 20 \\ 1 & 25 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 13.9 \\ 16.9 \\ 19.4 \\ 20.8 \\ 23.2 \\ 25.4 \end{bmatrix}$$

$$A \quad X = b$$

(a) $y = 14.38 + .44x$ (b) $y = 14.38 + .44(18)$

$= 22.3$

(c) $28 = 14.38 + .44x$

$13.62 = .44x$

$x \cong 31.0$ or <u>2006</u>

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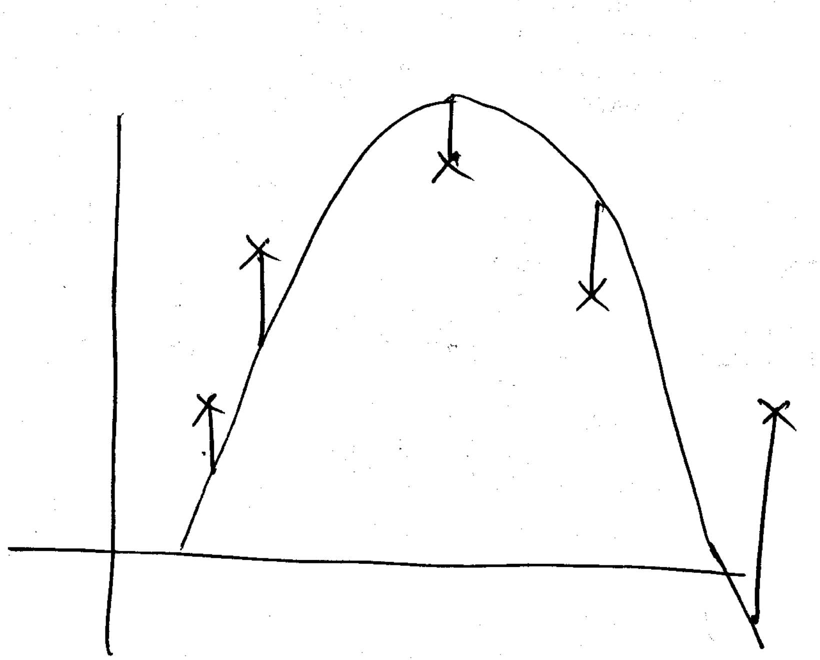
e.g. Exa 4.2, p 30 (notes)

x	1.82	3.69	4.88	2.29	5.23
y	2.03	4.56	3.73	3.28	2.94

x = time (sec)
y = height (m)



Least squares idea: Find $y = \underline{a_0} + \underline{a_1}x + \underline{a_2}x^2$ minimizing the ~~vert~~ square of vertical distance

$$a_0 + a_1(1.82) + a_2(1.82)^2 = 2.03$$

$$a_0 + a_1(3.69) + a_2(3.69)^2 = 4.56$$

$$a_0 + a_1(4.88) + a_2(4.88)^2 = 3.73$$

$$a_0 + a_1(2.29) + a_2(2.29)^2 = 3.28$$

$$a_0 + a_1(5.23) + a_2(5.23)^2 = 2.94$$

$$\begin{bmatrix} 1 & 1.82 & (1.82)^2 \\ 1 & 3.69 & (3.69)^2 \\ 1 & 4.88 & (4.88)^2 \\ 1 & 2.29 & (2.29)^2 \\ 1 & 5.23 & (5.23)^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2.03 \\ 4.56 \\ 3.73 \\ 3.28 \\ 2.94 \end{bmatrix}$$

$$A \quad X = b$$

$$\underbrace{A^T A}_{3\times 3} X = \underbrace{A^T b}_{3\times 1}$$

Solve system