Absorbing Stochastic Matrices

6-6-13

e.g.

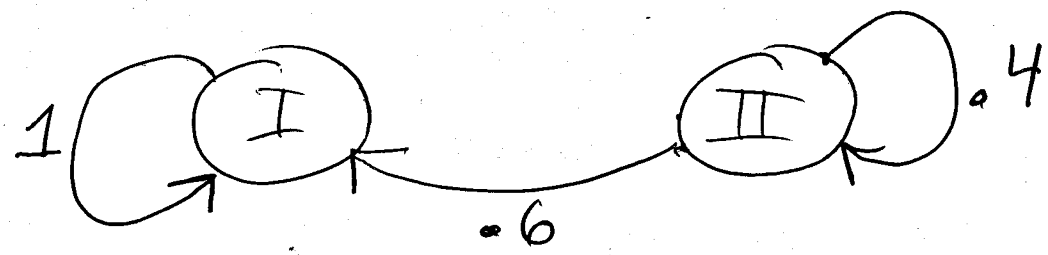
$$\begin{array}{c c} & \begin{array}{cc} I & II \end{array} \\ \begin{array}{c} I \\ II \end{array} & \begin{bmatrix} 1 & .6 \\ 0 & .4 \end{bmatrix} \end{array} = A$$

① Matrix is <u>not</u> regular.

A^n always looks like $\begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$

② The 1 in row 1, column 1 means that everything in state I stays in state I.

③ Think about long-term behavior.



In the long run, <u>everything</u> goes to state I.

What happens when we look at $A, A^2, A^3, \ldots$

$$\begin{bmatrix} 1 & .6 \\ 0 & .4 \end{bmatrix}, \begin{bmatrix} 1 & .6 \\ 0 & .4 \end{bmatrix}^2 = \begin{bmatrix} 1 & .84 \\ 0 & .16 \end{bmatrix}, \begin{bmatrix} 1 & .6 \\ 0 & .4 \end{bmatrix}^3 = \begin{bmatrix} 1 & .936 \\ 0 & .064 \end{bmatrix}$$

$$\begin{bmatrix} 1 & .6 \\ 0 & .4 \end{bmatrix}^4 = \begin{bmatrix} 1 & .9744 \\ 0 & .0256 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Absorbing Stochastic Matrix

① It has at least one absorbing state

② It is possible to get to one of the absorbing states from any other state.

what is an absorbing state?

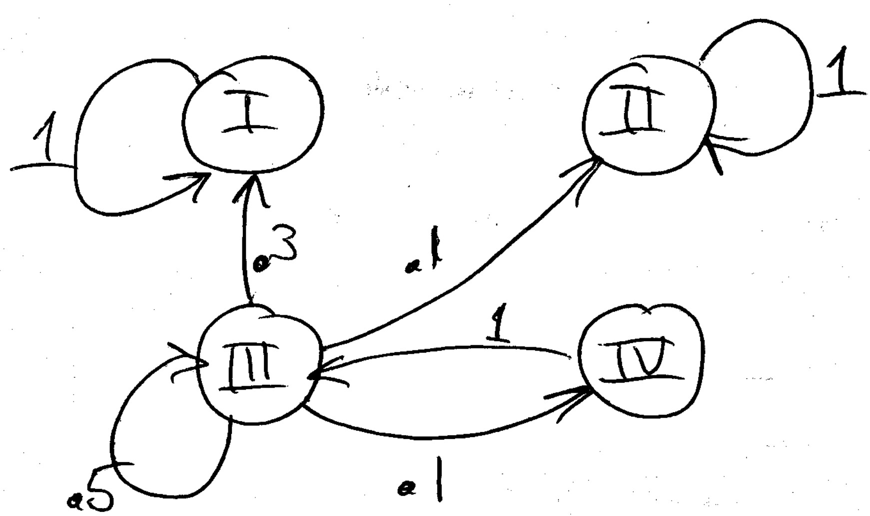
A state corresponding to a 1 on the main diagonal of transition matrix

e.g

	I	II	III	IV
I	1	0	.3	0
II	0	1	.1	0
III	0	0	.5	1
IV	0	0	.1	0

I, II absorbing

III, IV not absorbing



From each state there is a path to one of the absorbing states

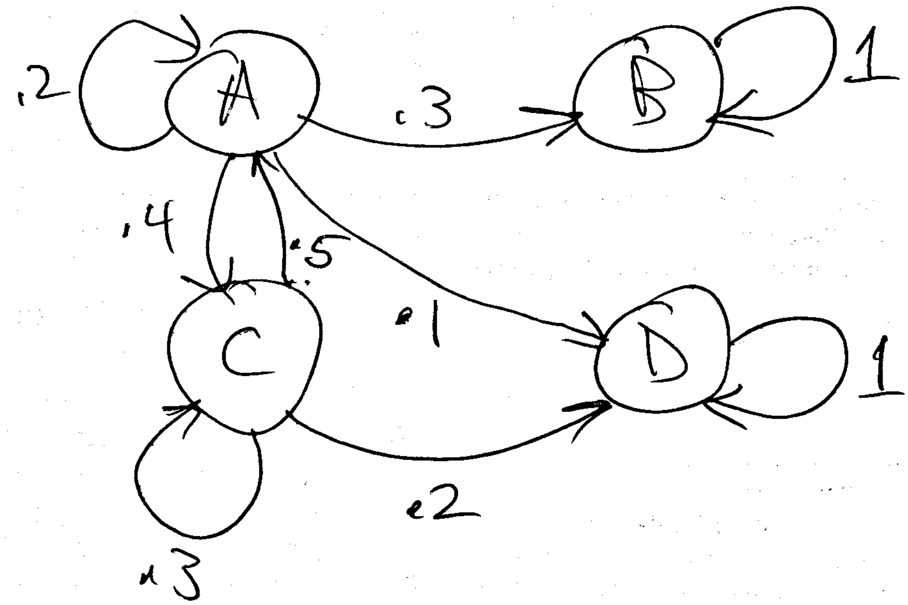
∴ A is an absorbing stochastic matrix.

eg 3

	A	B	C	D
A	.2	0	.5	0
B	.3	1	0	0
C	.4	0	.3	0
D	.1	0	.2	1

Absorbing? YES.

B, D are absorbing states.

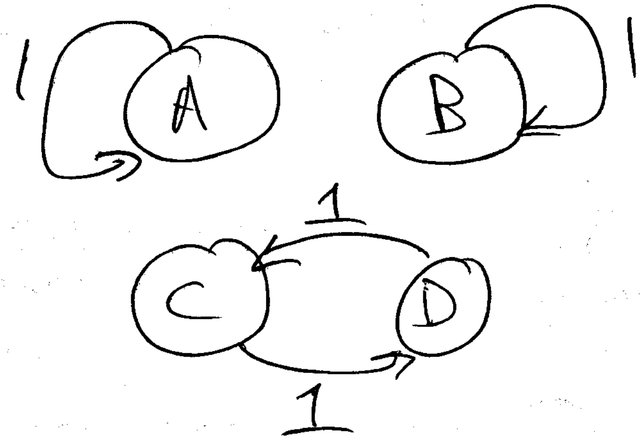


e.g.

	A	B	C	D
A	1	0	0	0
B	0	1	0	0
C	0	0	0	1
D	0	0	1	0

Absorbing? NO

A, B are absorbing.



Now the question is: Does an <u>absorbing</u> <u>stochastic matrix</u> exhibit long-term stability?

Standard form:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} .2 & 0 & .5 & 0 \\ .3 & 1 & 0 & 0 \\ .4 & 0 & .3 & 0 \\ .1 & 0 & .2 & 1 \end{bmatrix} \end{array} \longrightarrow \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} B & D & A & C \end{array} \\ \begin{bmatrix} 0 & 0 & .2 & .5 \\ 1 & 0 & .3 & 0 \\ 0 & 0 & .4 & .3 \\ 0 & 1 & .1 & .2 \end{bmatrix} \end{array}$$

$$\longrightarrow \begin{array}{c} \\ B \\ D \\ A \\ C \end{array} \begin{array}{c} \begin{array}{cccc} B & D & A & C \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & .3 & 0 \\ 0 & 1 & .1 & .2 \\ \hline 0 & 0 & .2 & .5 \\ 0 & 0 & .4 & .3 \end{array}\right] \end{array}$$

Could also have done:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} D & B & C & A \end{array} \\ \begin{bmatrix} 0 & 0 & .5 & .2 \\ 0 & 1 & 0 & .3 \\ 0 & 0 & .3 & .4 \\ 1 & 0 & .2 & .1 \end{bmatrix} \end{array} \longrightarrow \begin{array}{c} \\ D \\ B \\ C \\ A \end{array} \begin{array}{c} \begin{array}{cccc} D & B & C & A \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & .2 & .1 \\ 0 & 1 & 0 & .3 \\ \hline 0 & 0 & .3 & .4 \\ 0 & 0 & .5 & .2 \end{array}\right] \end{array}$$

So standard form looks like:

$$A = \begin{bmatrix} I & S \\ O & R \end{bmatrix}$$

(I: identity; O: all zeros; R: square)

In this case the stable matrix is

$$\begin{bmatrix} I & S(I-R)^{-1} \\ O & O \end{bmatrix}.$$

This means that if I look at $A, A^2, A^3, A^4, \ldots$

eg 5

	C	D	Sick
C	1	0	.7
D	0	1	.1
Sick	0	0	.2

$= A$

absorbing states? C, D.

stable matrix:

$$S = \begin{bmatrix} .7 \\ .1 \end{bmatrix} \qquad R = [.2]$$

$$I - R = [1] - [.2]$$

$$= [.8]$$

$$\left[\begin{array}{c|c} I & S(I-R)^{-1} \\ \hline 0 & 0 \end{array}\right]$$

$$(I-R)^{-1} = [.8]^{-1}$$

$$= \left[\frac{1}{.8}\right]$$

$$S(I-R)^{-1} = \begin{bmatrix} .7 \\ .1 \end{bmatrix} \left[\frac{1}{.8}\right] = \begin{bmatrix} .7/.8 \\ .1/.8 \end{bmatrix} = \begin{bmatrix} 7/8 \\ 1/8 \end{bmatrix}$$

Stable matrix is

	C	D	Sick
C	1	0	7/8
D	0	1	1/8
Sick	0	0	0

What does it mean?
$\frac{7}{8}$ of those sick eventually are cured
$\frac{1}{8}$ of those sick eventually die.

PP3
p429

$$\begin{array}{c|c|cc} & \text{I} & \text{II} & \text{III} \\ \hline \text{I} & 1 & .4 & 0 \\ \hline \text{II} & 0 & .2 & .1 \\ \text{III} & 0 & .4 & .9 \end{array}$$

$$S = \begin{bmatrix} .4 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} .2 & .1 \\ .4 & .9 \end{bmatrix}$$

$$I - R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .2 & .1 \\ .4 & .9 \end{bmatrix} = \begin{bmatrix} .8 & -.1 \\ -.4 & .1 \end{bmatrix}$$

$$(I-R)^{-1} = \frac{1}{.04}\begin{bmatrix} .1 & .1 \\ .4 & .8 \end{bmatrix} = \begin{bmatrix} 2.5 & 2.5 \\ 10 & 20 \end{bmatrix}$$

$$D = (.8)(.1) - (-.1)(-.4) = .08 - .04 = .04$$

$$S(I-R)^{-1} = \begin{bmatrix} .4 & 0 \end{bmatrix}\begin{bmatrix} 2.5 & 2.5 \\ 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Stable matrix:

$$\begin{array}{c|c|cc} & \text{I} & \text{II} & \text{III} \\ \hline \text{I} & 1 & 1 & 1 \\ \hline \text{II} & 0 & 0 & 0 \\ \text{III} & 0 & 0 & 0 \end{array}$$