6-5-2013

Markov processes

①
$$\begin{array}{c|c} & \text{I} \quad \text{II} \quad \cdots \\ \begin{array}{c}\text{I}\\ \text{II}\\ \vdots\end{array} & \begin{bmatrix} \cdot & \cdot & \cdot \\ \otimes & \cdot & \cdot \\ & & \end{bmatrix} \end{array} = A$$

stochastic matrix (transition matrix)

e.g.: pct of objects in state I that go to state II,

② X = initial distribution. (write $[\ ]_0$)

distribution after 1 time period, $[\ ]_1$ is

$$[\ ]_1 = AX = A[\ ]_0$$

③ Stable distribution

$$A, A^2, A^3, \ldots, A^n \longrightarrow \begin{bmatrix} | & | & \cdots & | \end{bmatrix}$$

stable distribution

all columns are the same.

e.g. $\begin{bmatrix} .3 \\ .1 \\ .6 \end{bmatrix}$ ← Eventually, no matter what distribution you start with, 30% end up in state I, 10% in state 2, 60% in state 3.

Only regular stochastic matrices are guaranteed to have a stable distribution.

How to find the stable distribution?

$\underline{X}$ stable distribution implies:

$$A\underline{X} = \underline{X}$$

e.g. $\begin{bmatrix} .7 & .4 \\ .3 & .6 \end{bmatrix} = A$ Find stable distribution.

$$\underline{X} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} .7 & .4 \\ .3 & .6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} .7x + .4y &= x \\ .3x + .6y &= y \end{aligned} \longrightarrow \begin{aligned} -.3x + .4y &= 0 \\ .3x - .4y &= 0 \end{aligned}$$

$$\underbrace{.7x - x}_{\substack{(.7-1)x \\ -.3x}} + .4y = 0$$

$$\begin{bmatrix} -.3 & .4 & 0 \\ .3 & -.4 & 0 \end{bmatrix} \cdot \frac{-1}{.3}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -4/3 & 0 \\ .3 & -.4 & 0 \end{bmatrix}$$

$$-\frac{3}{10}\begin{bmatrix} 1 & -4/3 & 0 \\ \frac{3}{10} & -\frac{2}{5} & 0 \end{bmatrix} \to \left[\begin{array}{cc|c} 1 & -4/3 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \underline{\underline{0=0}}$$

$$\frac{\cancel{3}}{\cancel{10}_5}\cdot\frac{\cancel{4}^2}{\cancel{3}} = \frac{2}{5} \qquad x - \frac{4}{3}y = 0 \qquad x = \frac{4}{3}y$$

$y =$ any value

Which value of y do we pick?

We also know $\underline{x+y=1}$

$$x+y=1 \to y = 1-x \to x = \frac{4}{3}(1-x)$$

$$x = \frac{4}{3} - \frac{4}{3}x \to \frac{7}{3}x = \frac{4}{3} \to x = \frac{4}{3}\cdot\frac{3}{7} = \frac{4}{7}$$

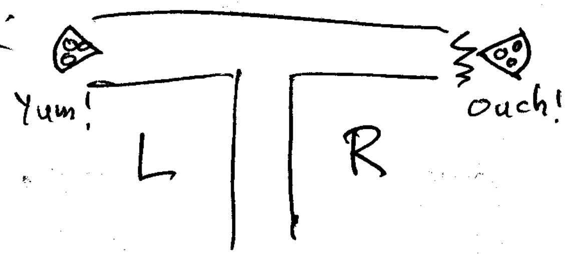
$$\left.\begin{array}{l} x = \frac{4}{7} \\ y = \frac{3}{7} \end{array}\right] \text{ stable distribution}$$

To find stable distribution

solve: ① sum of entries in $X = 1$

② $AX = X$.

e.g.



$$\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{bmatrix} .9 & .6 \\ .1 & .4 \end{bmatrix} \end{array} = A$$

what is stable distribution?

Solve:
$$x + y = 1 \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\left.\begin{array}{l} .9x + .6y = x \\ .1x + .4y = y \end{array}\right\} \rightarrow AX = X$$

$$\downarrow$$

$$\begin{aligned} x + y &= 1 \\ -.1x + .6y &= 0 \\ .1x - .6y &= 0 \end{aligned}$$

$$-\frac{1}{10}\begin{bmatrix} 1 & 1 & 1 \\ -\frac{1}{10} & \frac{3}{5} & 0 \\ \frac{1}{10} & -\frac{3}{5} & 0 \end{bmatrix} \rightarrow 1\begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{7}{10} & \frac{1}{10} \\ 0 & -\frac{7}{10} & -\frac{1}{10} \end{bmatrix}$$

$$\left(\frac{10}{7}\right)\begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{7}{10} & \left(\frac{1}{10}\right) \\ 0 & 0 & 0 \end{bmatrix} \rightarrow -1\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{6}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} 6/7 \\ 1/7 \end{bmatrix}$$

<u>eg</u> #10)
p 420

$$\begin{bmatrix} .3 & .1 & .2 \\ .4 & .8 & .6 \\ .3 & .1 & .2 \end{bmatrix} = A$$

Find stable dist.

$$\underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

solve

$$x + y + z = 1$$

$$A\underline{X} = \underline{X} \longrightarrow A\underline{X} - \underline{X} = 0$$

$$\longrightarrow (A - I)\underline{X} = 0$$

$$A - I = \begin{bmatrix} -.7 & .1 & .2 \\ .4 & -.2 & .6 \\ .3 & .1 & -.8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} \\ 10 \\ 10 \\ 10 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -.7 & .1 & .2 & 0 \\ .4 & -.2 & .6 & 0 \\ .3 & .1 & -.8 & 0 \end{array}\right]$$

can eliminate this at start.

$$\begin{array}{c} -3 \\ \cancel{-4}\,7 \end{array}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -7 & 1 & 2 & 0 \\ 4 & -2 & 6 & 0 \\ 3 & 1 & -8 & 0 \end{bmatrix} \rightarrow \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 8 & 9 & 7 \\ 0 & -6 & 2 & -4 \\ 0 & -2 & -11 & -3 \end{bmatrix}$$

$$2\,\cancel{6}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 9/8 & 7/8 \\ 0 & -6 & 2 & -4 \\ 0 & -2 & -11 & -3 \end{bmatrix} \rightarrow \frac{4}{35}\begin{bmatrix} 1 & 0 & -\frac{1}{8} & \frac{1}{8} \\ 0 & 1 & 9/8 & 7/8 \\ 0 & 0 & \frac{35}{4} & 5/4 \\ 0 & 0 & -35/4 & -5/4 \end{bmatrix}$$

get rid of this.

$$\frac{9}{\cancel{8}_4}\cdot\cancel{6}^3 + 2 = \frac{27}{4} + \frac{8}{4} = \frac{35}{4}$$

$$\frac{7}{\cancel{8}_4}\cdot\cancel{6}^3 - 4 = \frac{21}{4} - \frac{16}{4} = \frac{5}{4}$$

$$\frac{9}{\cancel{8}_4}\cdot\cancel{2} - 11 = \frac{9}{4} - \frac{44}{4} = \frac{-35}{4}$$

$$\frac{7}{\cancel{8}_4}\cdot\cancel{2} - 3 = \frac{7}{4} - \frac{12}{4} = -\frac{5}{4}$$

$$-\frac{9}{8}\begin{bmatrix} 1 & 0 & -\frac{1}{8} & \frac{1}{8} \\ 0 & 1 & 9/8 & 7/8 \\ 0 & 0 & 1 & 1/7 \end{bmatrix}$$

last row ignored

$$\begin{bmatrix} 1 & 0 & 0 & 1/7 \\ 0 & 1 & 0 & 5/7 \\ 0 & 0 & 1 & 1/7 \end{bmatrix} \qquad X = \begin{bmatrix} 1/7 \\ 5/7 \\ 1/7 \end{bmatrix}$$

$$-\frac{9}{8}\cdot\frac{1}{7} + \frac{7}{8}\cdot\frac{7}{7} = \frac{-9}{56} + \frac{49}{56} = \frac{40}{56} = \frac{5}{7}$$

e.g. #30 p412)

$$A = \begin{array}{c|ccc} & CF & CN & TCT \\ \hline CF & .69 & .16 & .20 \\ CN & .12 & .74 & .14 \\ TCT & .19 & .10 & .66 \end{array}$$

Find stable matrix (MATLAB)

demo3.txt

with mistakes!

Problem 10 (MyMathLab Quiz 3)

$$A = \begin{bmatrix} .3 & .7 \\ .7 & .3 \end{bmatrix}$$

$$A - I = \begin{bmatrix} -.7 & .7 \\ .7 & -.7 \end{bmatrix}$$

$x + y = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ -.7 & .7 & 0 \\ .7 & -.7 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -.7 & .7 & 0 \end{bmatrix}$$

$$\downarrow$$

$$7\begin{bmatrix} 1 & 1 & 1 \\ -7 & 7 & 0 \end{bmatrix}$$

$$\frac{1}{14}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 14 & 7 \end{bmatrix} \leftarrow$$

$$\uparrow$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \end{bmatrix}$$

$y = \frac{1}{2}$

$\therefore x = \frac{1}{2}$

$$\bar{X} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$$

Problem 13

$$A = \begin{bmatrix} .6 & .7 & .2 \\ .1 & .2 & .5 \\ .3 & .1 & .3 \end{bmatrix} \qquad A - I = \begin{bmatrix} -.4 & .7 & .2 \\ .1 & -.8 & .5 \\ .3 & .1 & -.7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 7 & 2 & 0 \\ 1 & -8 & 5 & 0 \\ 3 & 1 & -7 & 0 \end{bmatrix} \quad \frac{-1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 7 & 2 & 0 \\ 1 & -8 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 11 & 6 & 4 \\ 0 & -9 & 4 & -1 \end{bmatrix} \quad \frac{9}{1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 6/11 & 4/11 \\ 0 & -9 & 4 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5/11 & 7/11 \\ 0 & 1 & 6/11 & 4/11 \\ 0 & 0 & 98/11 & 25/11 \end{bmatrix} \quad -\frac{6}{11} \begin{bmatrix} 1 & 0 & 5/11 & 7/11 \\ 0 & 1 & 6/11 & 4/11 \\ 0 & 0 & 1 & .25 \end{bmatrix}$$

$$9 \cdot \frac{6}{11} + 4 = \frac{54}{11} + \frac{44}{11} = \frac{98}{11}$$

$$9 \cdot \frac{4}{11} - 1 = \frac{36}{11} - \frac{11}{11} = \frac{25}{11}$$

$$\frac{25}{11} \cdot \frac{11}{98} = \frac{25}{98}$$

$$\begin{bmatrix} 1 & 0 & 0 & .52 \\ 0 & 1 & 0 & .23 \\ 0 & 0 & 1 & .25 \end{bmatrix}$$

8.3 Absorbing Stochastic Matrices

e.g

$$A = \begin{array}{c|cc} & \mathrm{I} & \mathrm{II} \\ \hline \mathrm{I} & 1 & .6 \\ \mathrm{II} & 0 & .4 \end{array}$$

Is A regular?

$$\begin{bmatrix} 1 & .6 \\ 0 & .4 \end{bmatrix}\begin{bmatrix} 1 & .6 \\ 0 & .4 \end{bmatrix} = \begin{bmatrix} 1 & .84 \\ 0 & .16 \end{bmatrix} = A^2$$

$$\begin{bmatrix} 1 & .84 \\ 0 & .16 \end{bmatrix}\begin{bmatrix} 1 & .6 \\ 0 & .4 \end{bmatrix} = \begin{bmatrix} 1 & .936 \\ 0 & .064 \end{bmatrix} = A^3$$

Matrix is <u>not</u> regular.

What does the 1 in row 1, column 1 mean?

Everything in State I stays in State I.

I is called an <u>absorbing state</u>.

Matrices with absorbing states still have predictable long-term behavior.