1.6. One-sided Limits and Continuity

One-sided Limit

If $f(x)$ approaches L as x tends toward c from the left ($x < c$), we write

$$\lim_{x \to c^-} f(x) = L.$$

Likewise, if $f(x)$ approaches M as x tends toward c from the right ($x > c$), then

$$\lim_{x \to c^+} f(x) = M.$$

Example

Find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$ for the function

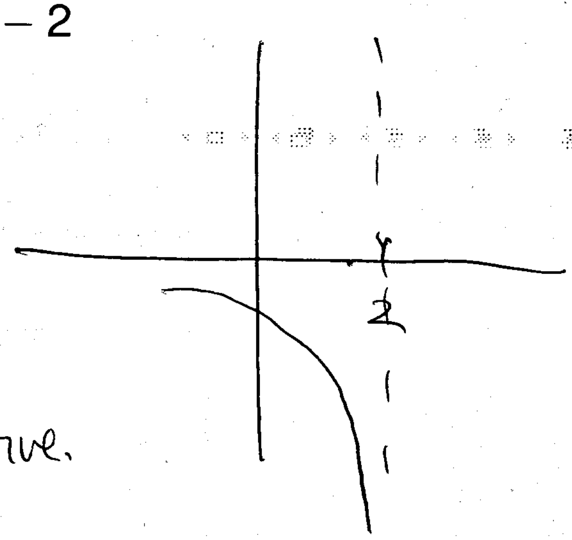
$$f(x) = \frac{x^2 + 3}{x - 2}$$

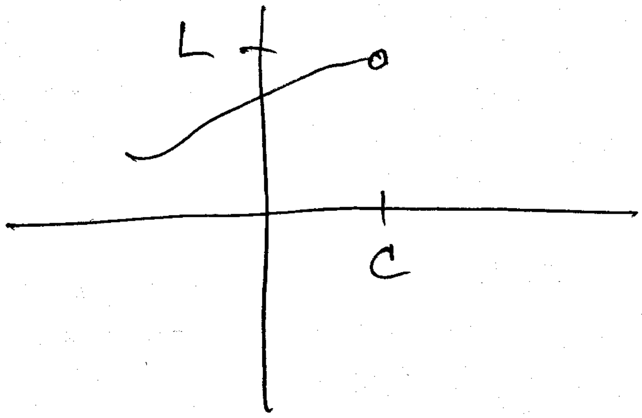
$$\lim_{x \to 2^-} \frac{x^2+3}{x-2} = -\infty$$

vertical asymptote

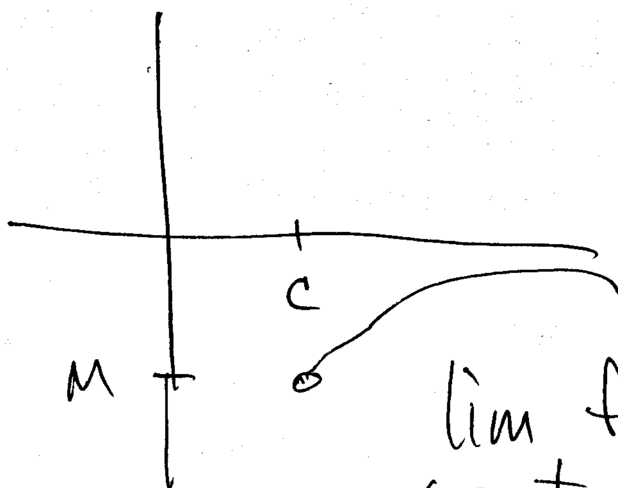
$x<2$ $\quad x^2+3 > 0 \quad \frac{(+)}{(-)} =$ negative.

$\quad\quad\quad x-2 < 0$

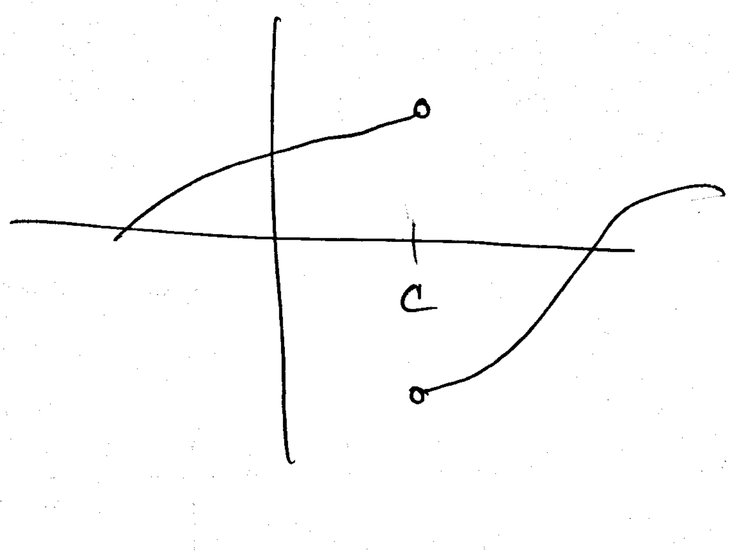




$$\lim_{x \to c^-} f(x) = L$$

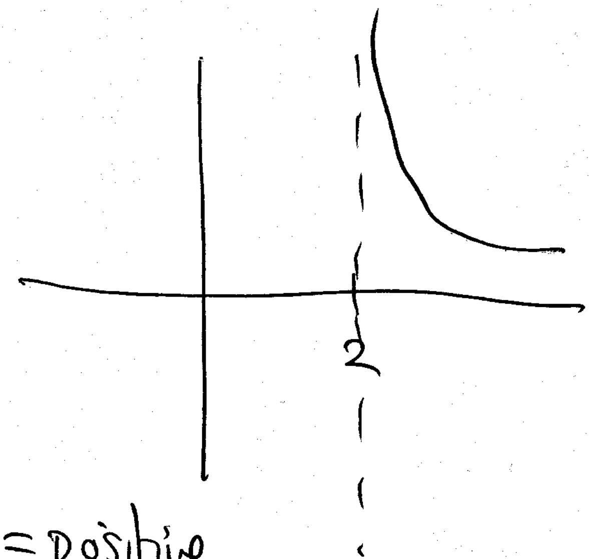


$$\lim_{x \to c^+} f(x) = L$$



$\lim_{x \to c} f(x)$ does not exist.

$$\lim_{x\to 2^+} \frac{x^2+3}{x-2} = +\infty$$

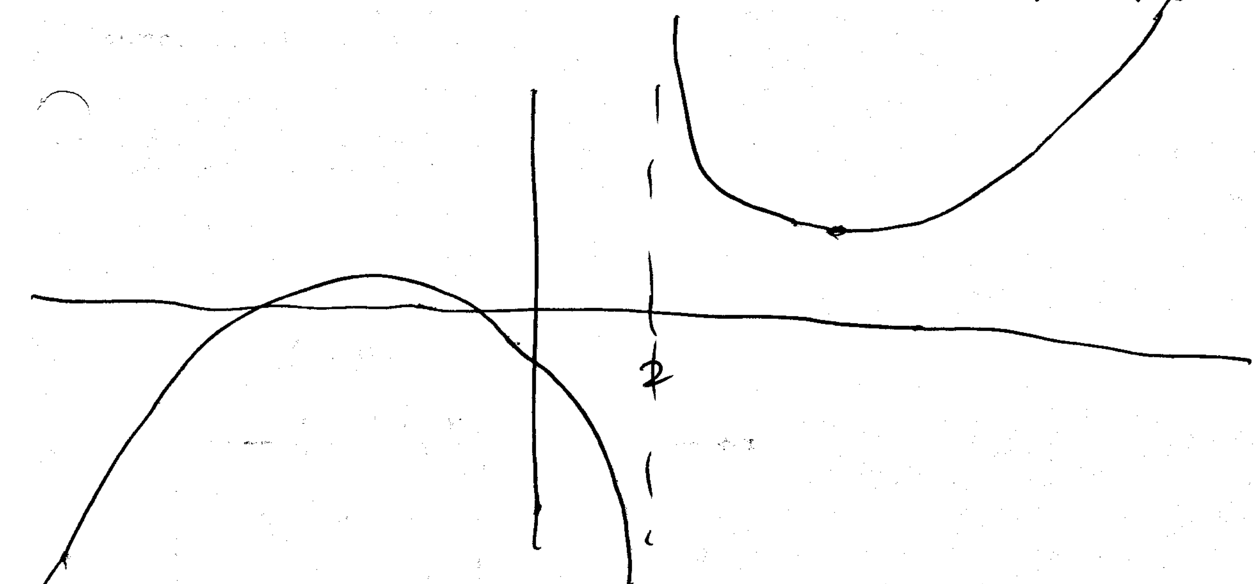


vertical asymptote

$x>2$ $\quad x^2+3>0$

$\quad\quad\quad x-2>0$ $\quad \frac{(+)}{(+)} = \text{positive}$

$$\lim_{x\to+\infty} \frac{x^2+3}{x-2} = \lim_{x\to+\infty} \frac{x^2}{x} = \lim_{x\to+\infty} x = +\infty.$$



$$\lim_{x\to-\infty} \frac{x^2+3}{x-2} = \lim_{x\to-\infty} x = -\infty$$

One-sided Limit

Example

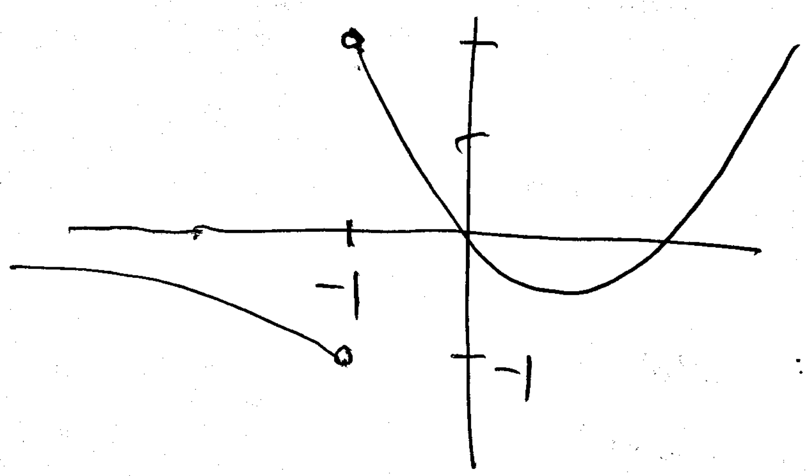
Find $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to -1^+} f(x)$ for the function

$$f(x) = \begin{cases} \dfrac{2}{x-1} & \text{if } x < -1 \\ x^2 - x & \text{if } x \geq -1 \end{cases}$$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{2}{x-1} = \frac{2}{-2} = -1$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x^2 - x) = (-1)^2 - (-1) = 2$$

$\lim_{x \to 1} f(x)$ DOES NOT EXIST


-1
-1

Existence of a Limit

Theorem

The two-sided limit $\lim_{x \to c} f(x)$ *exists if and only if the two one-sided limits* $\lim_{x \to c^-} f(x)$ *and* $\lim_{x \to c^+} f(x)$ *both exist and are equal, and then*

$$\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

Example

Determine whether $\lim_{x \to 1} f(x)$ exists, where

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ -x^2 + 2x + 2 & \text{if } x \geq 1 \end{cases}$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-x^2+2x+2) = 3$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x+1) = 3$$

$$\therefore \lim_{x \to 1} f(x) = 3$$

Continuity

Continuity

A function f is *continuous* at c if all three of these conditions are satisfied:

a. $f(c)$ is defined

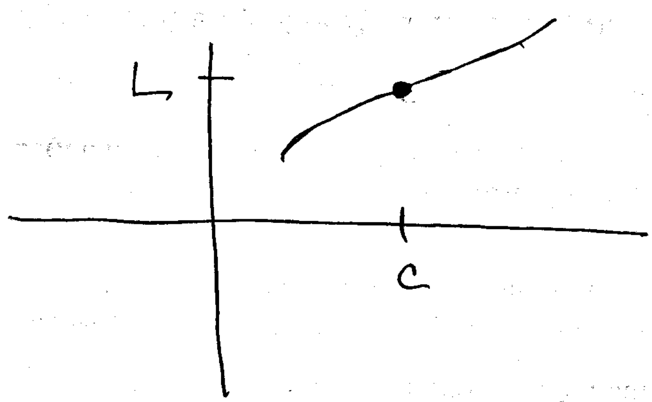
b. $\lim_{x \to c} f(x)$ exists $\qquad \lim_{x \to c} f(x) = L$

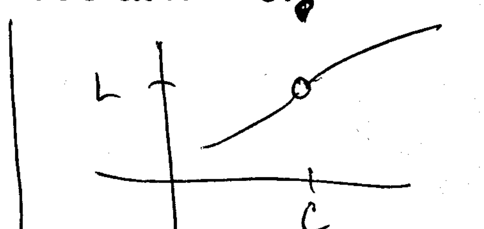
c. $\lim_{x \to c} f(x) = f(c)$

If $f(x)$ is not continuous at c, it is said to have a *discontinuity* there.

Example

Decide if $f(x) = x^3 - x^2 + x - 4$ is continuous at $x = 0$.





$\lim_{x \to c} f(x) = L \neq f(c)$

Continuity

Example

Decide if $f(x) = \dfrac{2x+5}{2x-4}$ is continuous at $x = 2$.

$f(2)$ not defined so NO

Continuity

Continuity of Polynomials and Rational Functions

A polynomial or a rational function is continuous *wherever it is defined*.

Example

List all values of x for which $f(x)$ is not continuous

$$f(x) = \frac{x^2 - 2x + 1}{x^2 - x - 2}$$

Continuity

Example

Decide if $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x-1 & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$. NO

$f(0) = -1$

$\lim_{x \to 0} f(x)$ DNE

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x-1 = -1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x+1 = 1$

Continuity

Example

Find the value of the constant A such that the function

$$f(x) = \begin{cases} 1 - 2x & \text{if } x < 2 \\ Ax^2 + 2x - 3 & \text{if } x \geq 2 \end{cases}$$

will be continuous for all x.

Intermediate Value Property

The intermediate value property

If $f(x)$ is continuous on the interval $a \leq x \leq b$ and L is a number between $f(a)$ and $f(b)$, the $f(c) = L$ for some number c between a and b. In other words, a continuous function attains all values between any two of its values.

Example

Show that the equation $\sqrt[3]{x} = x^2 + 2x - 1$ must have at least one solution on the interval $0 \leq x \leq 1$.