Exponential & Logarithmic Functions

Derivative $\quad y = f(x) \qquad \frac{dy}{dx} = f'(x)$

So far we have dealt with

polynomials $\qquad 6x^3 + 10x^2 - 5x + 3$

rational functions $\qquad \dfrac{2x^2+3}{x^3+x}$

(fractional) powers $\qquad x^{3/2} \quad (x^2+3)^{-5/2}$

etc.

Now we will add to our list new functions: $\quad y = 3^x \qquad y = \log_5(x)$

4.1 Exponential Functions

Definition of b^n for rational values of n (and $b > 0$)

¿

- Integer Powers: If n is a positive integer,

$$b^n = \underbrace{b \cdot b \cdots b}_{n \text{ factors}}$$

- Fractional Powers: If n and m are positive integers,

$$b^{n/m} = (\sqrt[m]{b})^n = \sqrt[m]{b^n}$$

- Negative Powers: $b^{-n} = \dfrac{1}{b^n}$
- Zero Power: $b^0 = 1$

$$\boxed{2^{3/7} \approx 1.3459}$$

$$(2^3)^{\frac{1}{7}} = 8^{\frac{1}{7}} = 2^{3/7} \qquad (2^{1/7})^3 = 2^{3/7}$$

$$2^{\sqrt{2}} \approx$$

$\sqrt{2} \approx 1.4142\ldots$

$$\begin{array}{ll}
1.4 & 2^{1.4} = 2^{\frac{14}{10}} = 2^{7/5} \approx 2.6390 \\
1.41 & 2^{1.41} = 2^{\frac{141}{100}} \approx 2.6574 \\
1.414 & 2^{1.414} = 2^{\frac{1414}{1000}} \approx 2.6647 \\
\lim \downarrow & 2^{1.4142} = 2^{\frac{14142}{10000}} \approx 2.6651 \\
\sqrt{2} & \\
 & \lim \downarrow \\
 & 2^{\sqrt{2}} = 2.665144143\ldots
\end{array}$$

Exponential Functions

Definition

If b is a positive number other than 1 ($b > 0, b \neq 1$), there is a unique function called the exponential function with base b that is defined by

$$f(x) = b^x \quad \text{for all real number } x$$

Example

Sketch the graphs of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$.

Think: Why $b > 0$? Say $b = -2$

$b^x = (-2)^x$ $x = 2$

$(-2)^2 = 4$

$x = 3$

$(-2)^3 = -8$

$(-2)^{2/6} \longrightarrow (-2)^{1/3} = \sqrt[3]{-2}$

$= \left(\sqrt[6]{-2}\right)^2$

not defined

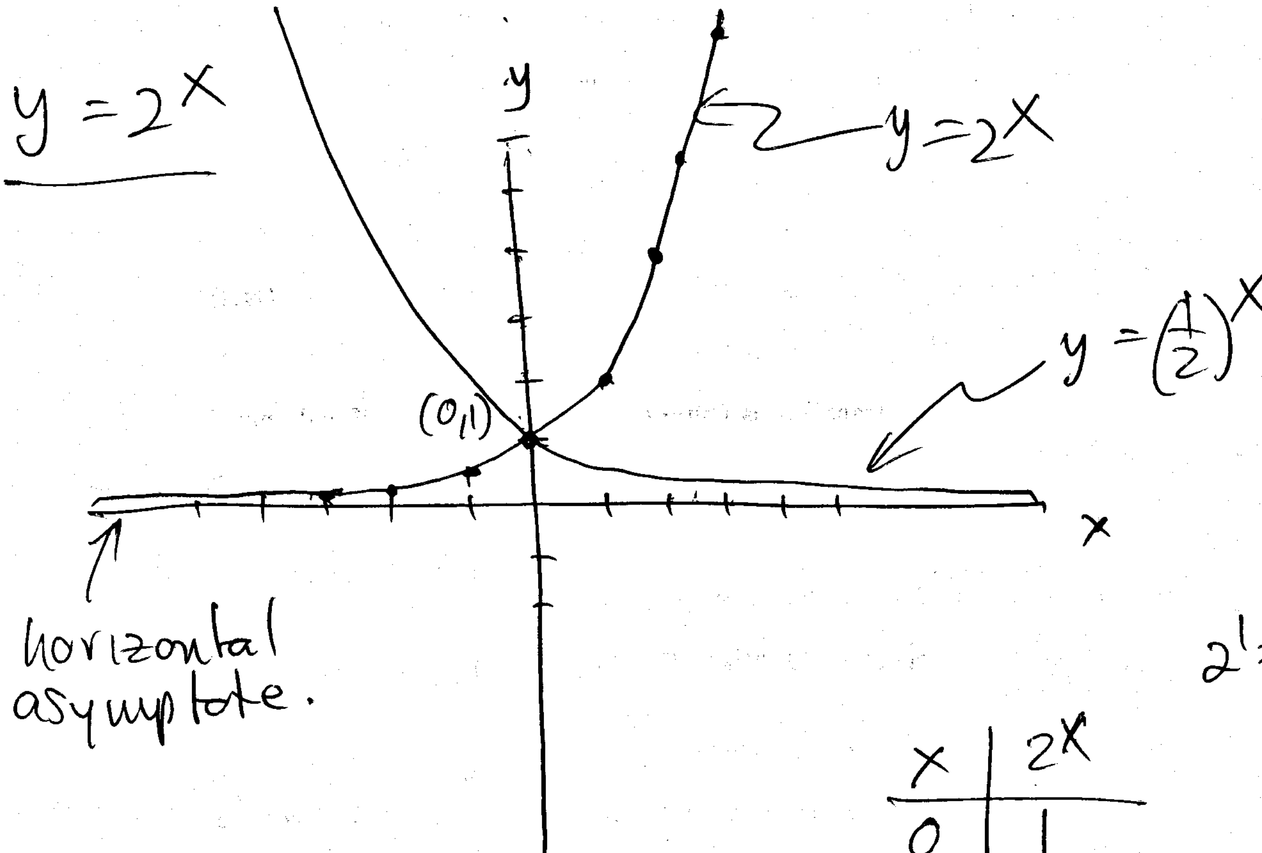
eg $\sqrt[3]{-8} = -2$ ok.

$(-2)^{1/2} = \sqrt{-2}$ not real.

why $b \neq 1$?

$1^x = 1$ $\quad$ $b=1$ then $b^x = 1$ all x.

$y = 2^x$



$2^1 = 2$

x	2^x
0	1
1	2
2	$2^2=4$
3	$2^3=8$
2.5	5.66
-1	$2^{-1}=\frac{1}{2}$
-2	$2^{-2}=\frac{1}{4}$
-3	$2^{-3}=\frac{1}{8}$

$y = \left(\frac{1}{2}\right)^x$

$= (2^{-1})^x$

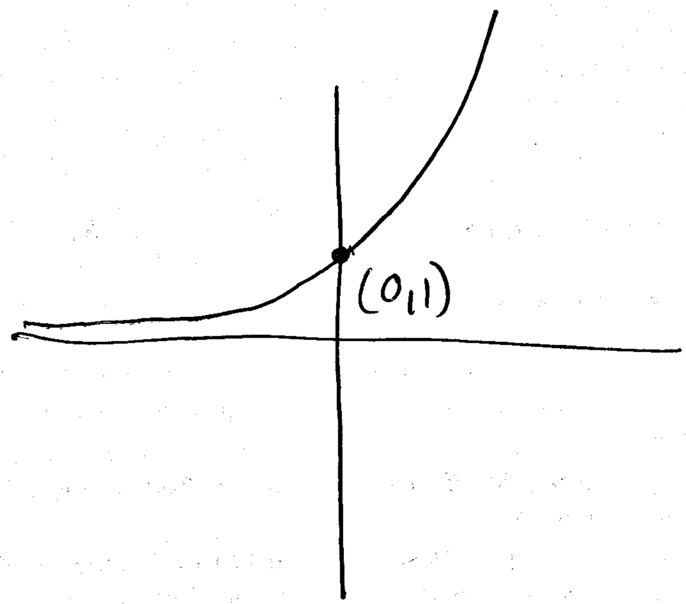
$= 2^{-x}$

x	$\left(\frac{1}{2}\right)^x$
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$
-1	$\left(\frac{1}{2}\right)^{-1}=2$
-2	$\left(\frac{1}{2}\right)^{-2}=4$
-3	$\left(\frac{1}{2}\right)^{-3}=8$

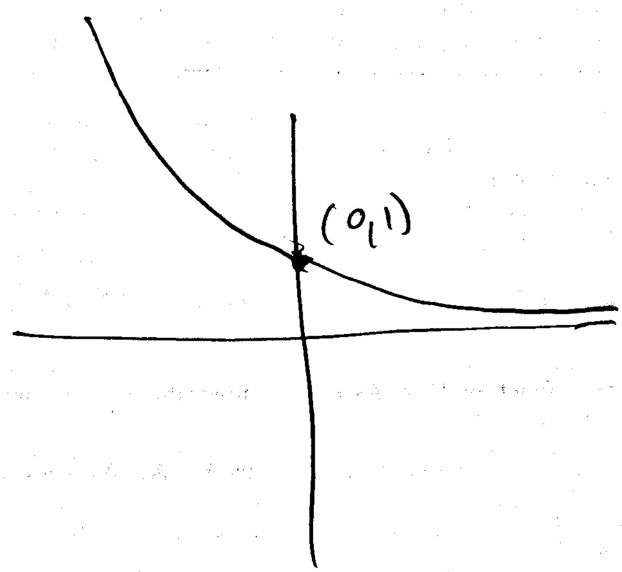
In general:

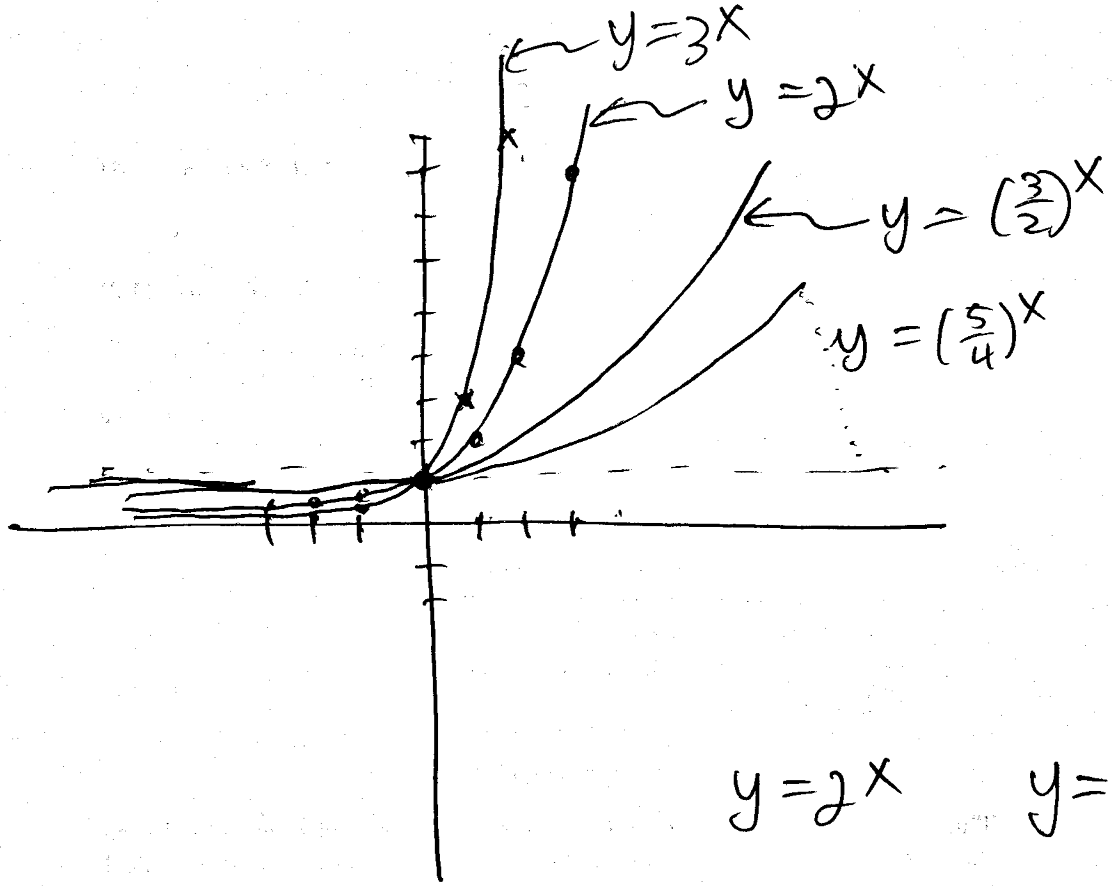
$f(x) = b^x \quad \cancel{b \neq 0} \quad b \neq 1,\ b > 0$

If $b > 1$



If $0 < b < 1$





$y=2^x$ $\quad y=\left(\frac{3}{2}\right)^x$

$y=3^x$

x	3^x
1	3
-1	$\frac{1}{3}$
-2	$\frac{1}{9}$

Exponential Functions

Basic Properties of Exponential Functions

For bases a, b and any real numbers x, y, we have

- The equality rule: $b^x = b^y$ if and only if $x = y$
- The product rule: $b^x b^y = b^{x+y}$
- The quotient rule: $\dfrac{b^x}{b^y} = b^{x-y}$
- The power rule: $(b^x)^y = b^{xy}$
- The multiplication rule: $(ab)^x = a^x b^x$
- The division rule: $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Exponential Functions

Example

Evaluate the given expression.

a. $8^{2/3}$ $= (8^{1/3})^2 = 2^2 = 4$

b. $(4^{2/3})(2^{2/3})$ $= (4 \cdot 2)^{2/3} = 8^{2/3} = 4$

c. $\dfrac{(3^{1.3})(3^{2.5})}{3^{3.2}}$ $= 3^{1.3+2.5-3.2} = 3^{.6} = 3^{6/10} = 3^{3/5}$

d. $(x^{3/2})^{-4/3}$ $= x^{\left(\frac{3}{2}\right)\left(-\frac{4}{3}\right)} = x^{-2}$

$3^{5/2} \cdot 7^{2/3}$ NOT SO EASY

NEED COMMON BASE
OR COMMON EXPONENT

Exponential Functions

Example

Find all real numbers x that satisfy the given equation.

a. $3^x 2^{2x} = 144$

Use: $b^x = b^y \Rightarrow x = y$

$3^x \cdot (2^2)^x = 3^x \cdot 4^x = 12^x$

$12^x = 144$ $\quad x = 2$ $\quad 12^x = 12^2$

b. $2^{3-x} = 4^x$

$2^{3-x} = 2^{2x}$ $\qquad 4^x = (2^2)^x = 2^{2x}$

$3 - x = 2x$

$x = 1$.