Quiz 6 Wednesday ← Sec 2.6

e.g. $x + \frac{1}{y} = 4$ Find $\frac{dy}{dx}$

$$\frac{d}{dx}\left(x + \frac{1}{y}\right) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{1}{y}\right) = \frac{d}{dx}(4)$$

Think of y as a function of x.

$$\frac{d}{dx}\left(\frac{1}{y}\right) = \frac{d}{dx}\left(\frac{1}{h(x)}\right)$$

$$= \frac{d}{dx}[h(x)]^{-1}$$

$$= (-1)[h(x)]^{-2}h'(x)$$

$$1 + \frac{-1}{y^2}\frac{dy}{dx} = 0$$

solve for this

$$1 = \frac{1}{y^2}\frac{dy}{dx}$$

$$\boxed{y^2 = \frac{dy}{dx}}$$

$$\frac{d}{dx}\left(\frac{1}{y}\right) = \frac{d}{dx}(y^{-1})$$

$$= (-1)y^{-2}\frac{dy}{dx}$$

$$= \frac{-1}{y^2}\frac{dy}{dx}$$

Implicit Differentiation

Example

Find $\frac{dy}{dx}$ if $4x - x^3y^2 = 2y$.

$$\frac{d}{dx}(4x - x^3y^2) = \frac{d}{dx}(2y)$$

$$\frac{d}{dx}(4x) - \frac{d}{dx}(x^3y^2) = \frac{d}{dx}(2y)$$

$$4 - \left[x^3\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x^3)\right] = 2\frac{dy}{dx}$$

$$4 - x^3 \cdot 2y\frac{dy}{dx} + y^2 \cdot 3x^2 = 2\frac{dy}{dx}$$

$$4 + 3x^2y^2 = 2\frac{dy}{dx} + 2x^3y\frac{dy}{dx}$$

$$4 + 3x^2y^2 = \frac{dy}{dx}(2 + 2x^3y)$$

$$\frac{dy}{dx} = \frac{4 + 3x^2y^2}{2 + 2x^3y} \quad //$$

Implicit Differentiation

Example

Find the equation of the tangent line to the curve
$x^2y^2 - 3xy = 5x + y + 1$ at the point $(0, -1)$.

Slope: Find $\frac{dy}{dx}$.

$$\frac{d}{dx}(x^2y^2 - 3xy) = \frac{d}{dx}(5x + y + 1)$$

$$\frac{d}{dx}(x^2y^2) - 3\frac{d}{dx}(xy) = 5 \cdot \frac{d}{dx}(x) + \frac{d}{dx}(y) + \frac{d}{dx}(1)$$

$$x^2\frac{d}{dx}(y^2) + y^2 \cdot 2x - 3\left[x \cdot \frac{d}{dx}(y) + y\right] = 5 + \frac{dy}{dx}$$

$$x^2 \cdot 2y\frac{dy}{dx} + 2xy^2 - 3x\frac{dy}{dx} - 3y = 5 + \frac{dy}{dx}$$

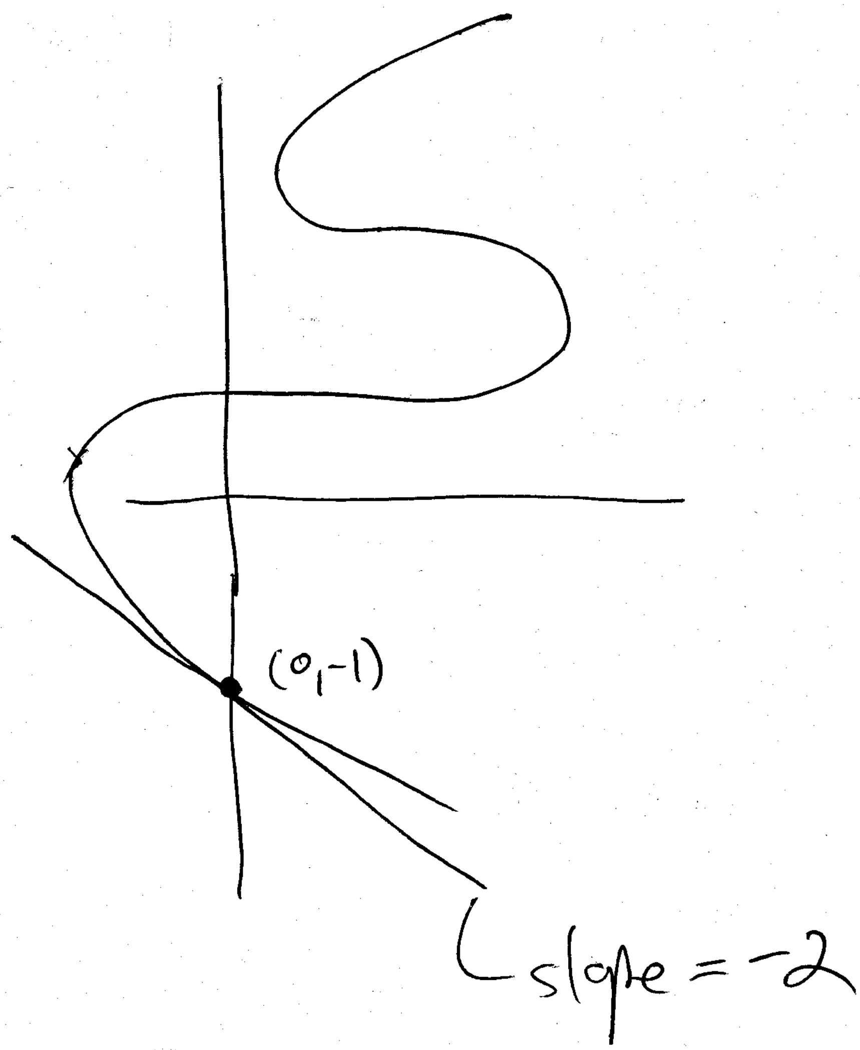
$$2xy^2 - 3y - 5 = \frac{dy}{dx} + 3x\frac{dy}{dx} - 2x^2y\frac{dy}{dx}$$

$$2xy^2 - 3y - 5 = \frac{dy}{dx}(1 + 3x - 2x^2y)$$

$$\frac{dy}{dx} = \frac{2xy^2 - 3y - 5}{1 + 3x - 2x^2y} \qquad \left.\frac{dy}{dx}\right|_{(0,-1)} = \frac{3-5}{1} = -2$$

Tangent line: $y+1=-2(x-0)$

$$\underline{y=-2x-1}\ //$$

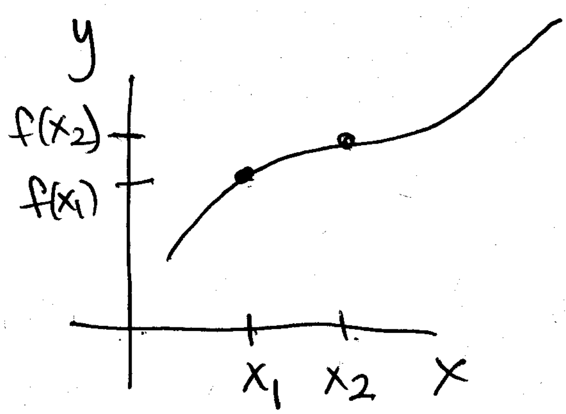


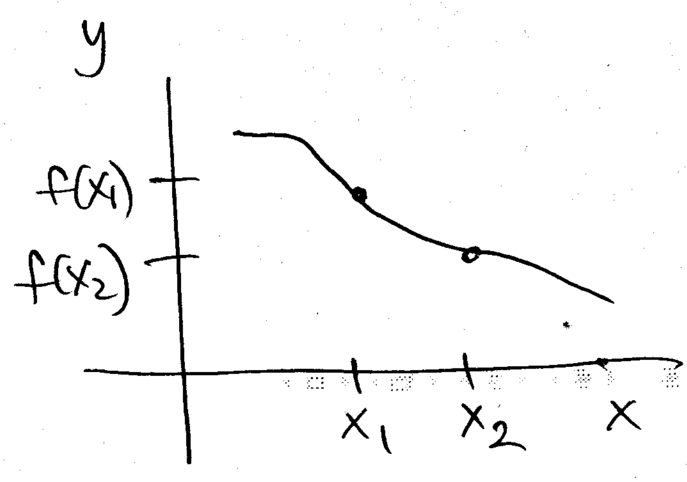
3.1. Increasing and Decreasing Functions; Relative Extrema

Increasing and Decreasing Functions

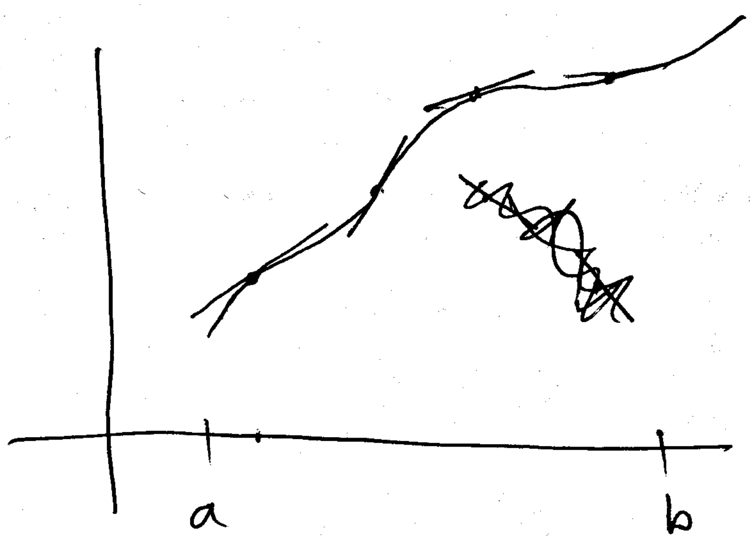
Let $f(x)$ be a function defined on the interval $a < x < b$, and let x_1 and x_2 be two numbers in the interval. Then

- $f(x)$ is increasing on the interval if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.
- $f(x)$ is decreasing on the interval if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$.





Idea: If $f'(x) > 0$ on some interval, then $f(x)$ is increasing on that interval.



If $f'(x) < 0$ on some interval then $f(x)$ is decreasing on that interval

Intervals of Increase and Decrease

Procedure for using the derivative to determine intervals of increase and decrease

Step 1. Find all values of x for which $f'(x) = 0$ or $f'(x)$ is not
defined ~~continuous~~, and mark these numbers on a number line. This divides the line into a number of open intervals.

Step 2. Choose a test number c from each interval $a < x < b$ determined in Step 1 and evaluate $f'(c)$. Then

- If $f'(c) > 0$, $f(x)$ is increasing on $a < x < b$.
- If $f'(c) < 0$, $f(x)$ is decreasing on $a < x < b$.

Intervals of Increase and Decrease

Example

Find the intervals of increase and decrease for the function

$$f(x) = 2x^5 - 5x^4 - 10x^3 + 7$$

$f'(x) = 10x^4 - 20x^3 - 30x^2$

Find where $f'(x) = 0$

$10x^4 - 20x^3 - 30x^2 = 0$

$10x^2(x^2 - 2x - 3) = 0$

$10x^2(x+1)(x-3) = 0$

$x = 0 \quad x = -1 \quad x = 3 \longleftarrow$ critical numbers

$+ + + + + 0 - - - 0 - - - - - - 0 + + + + +$

$f'(x)$ —— $-2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

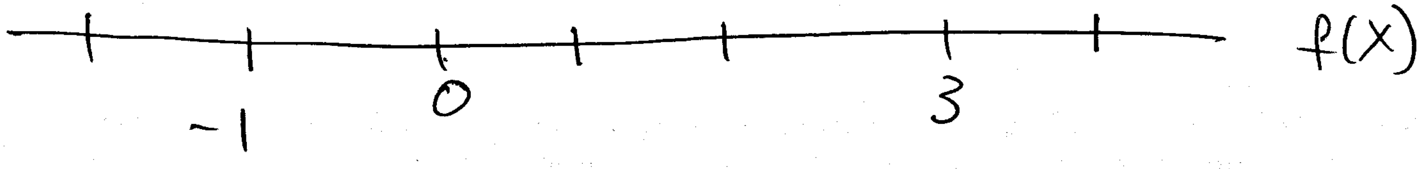
$f'(-2) = (10 \cdot 4)(-2+1)(-2-3) = 40(-1)(-5) > 0$

$f'(-\frac{1}{2}) = (\frac{10}{4})(\frac{1}{2})(-\frac{7}{2}) < 0$

$f'(1) = (10)(2)(-2) < 0$

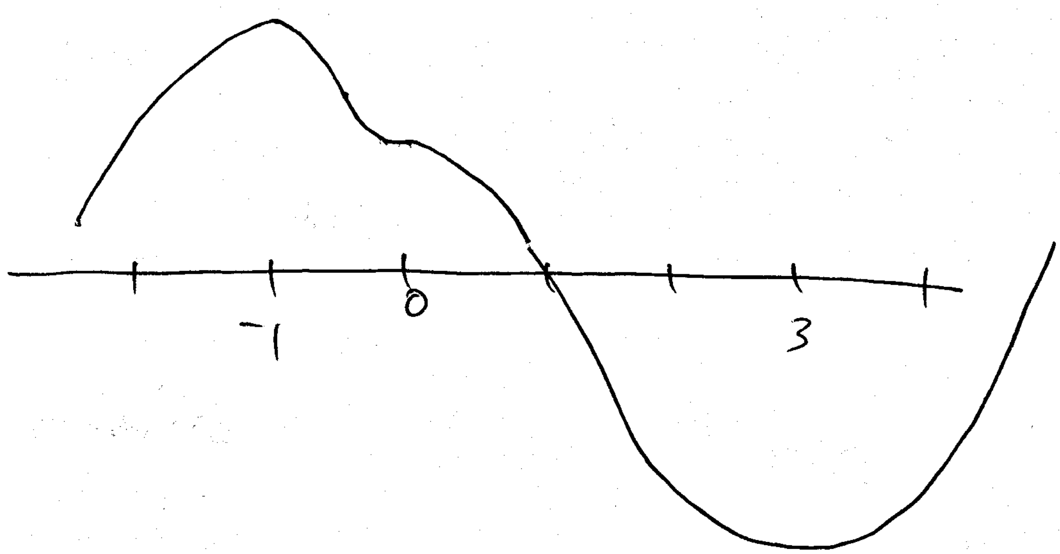
$f'(4) = (160)(5)(1) > 0$

increasing o decr. o decreasing o increasing



or $f(x)$ increasing on $(-\infty,-1) \cup (3,\infty)$

decreasing on $(-1,0) \cup (0,3)$



Intervals of Increase and Decrease

Example

Find the intervals of increase and decrease for the function

$$F(x) = \frac{x^2}{x-3}.$$

$$F'(x) = \frac{(x-3)(2x)-(x^2)(1)}{(x-3)^2} = \frac{2x^2-6x-x^2}{(x-3)^2}$$

$$= \frac{x^2-6x}{(x-3)^2} = \frac{x(x-6)}{(x-3)^2}$$

Solve $F'(x)=0$ $\quad x=0 \quad x=6$

$F'(x)$ undefined $\quad x=3$

← critical numbers

$+ + + + \; 0 \; - - - - \; \bullet \; - - - - \; 0 \; + + + +$

$F'(x)$

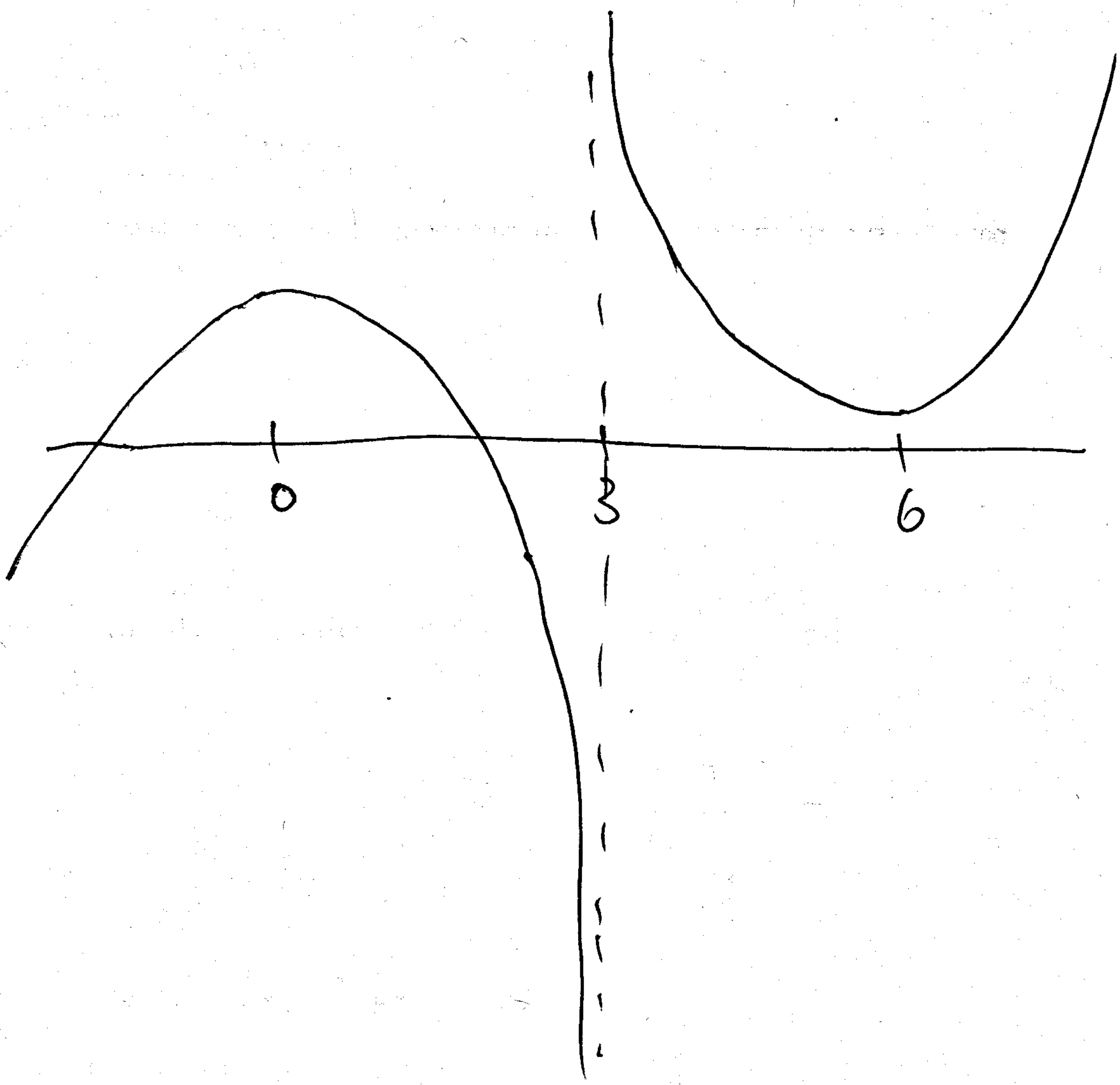
$-1 \quad 0 \quad 1 \quad 3 \quad 4 \quad 6$

$$F'(-1) = \frac{(-1)(-7)}{(-4)^2} > 0 \qquad F'(1) = \frac{(1)(-5)}{(-2)^2} < 0$$

$$F'(4) = \frac{(4)(-2)}{(1)^2} < 0 \qquad F'(7) = \frac{(7)(1)}{(4)^2} > 0$$

increasing decreasing decreasing increasing

$F(x)$ ——— 0 ——— 3 ——— 6 ———



$F(x)$ is increasing on $(-\infty, 0) \cup (6, \infty)$

decreasing on $(0, 3) \cup (3, 6)$

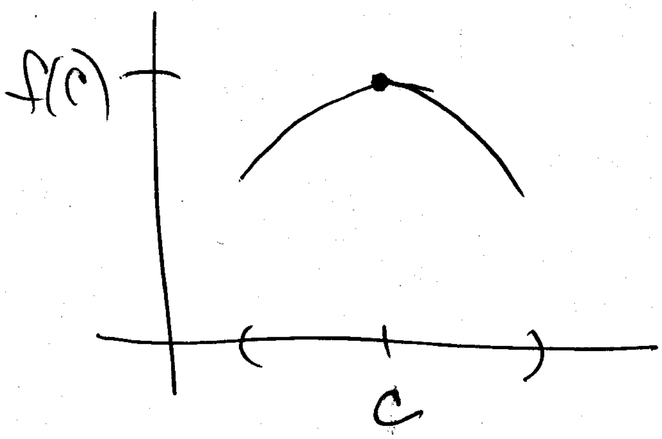
Relative Extrema (Maxima or Minima)

Definition

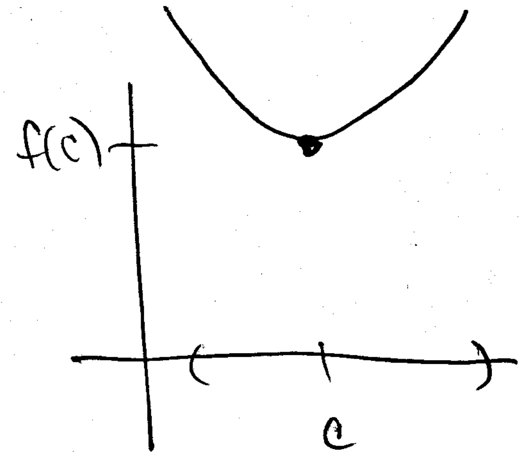
- The graph of the function $f(x)$ is said to have a relative maximum at $x = c$ if $f(c) \geq f(x)$ for all x in an interval $a < x < b$ containing c.
- Similarly, the graph has a relative minimum at $x = c$ if $f(c) \leq f(x)$ on such an interval.
- Collectively, the relative maxima and minima of f are called its relative extrema.

Definition

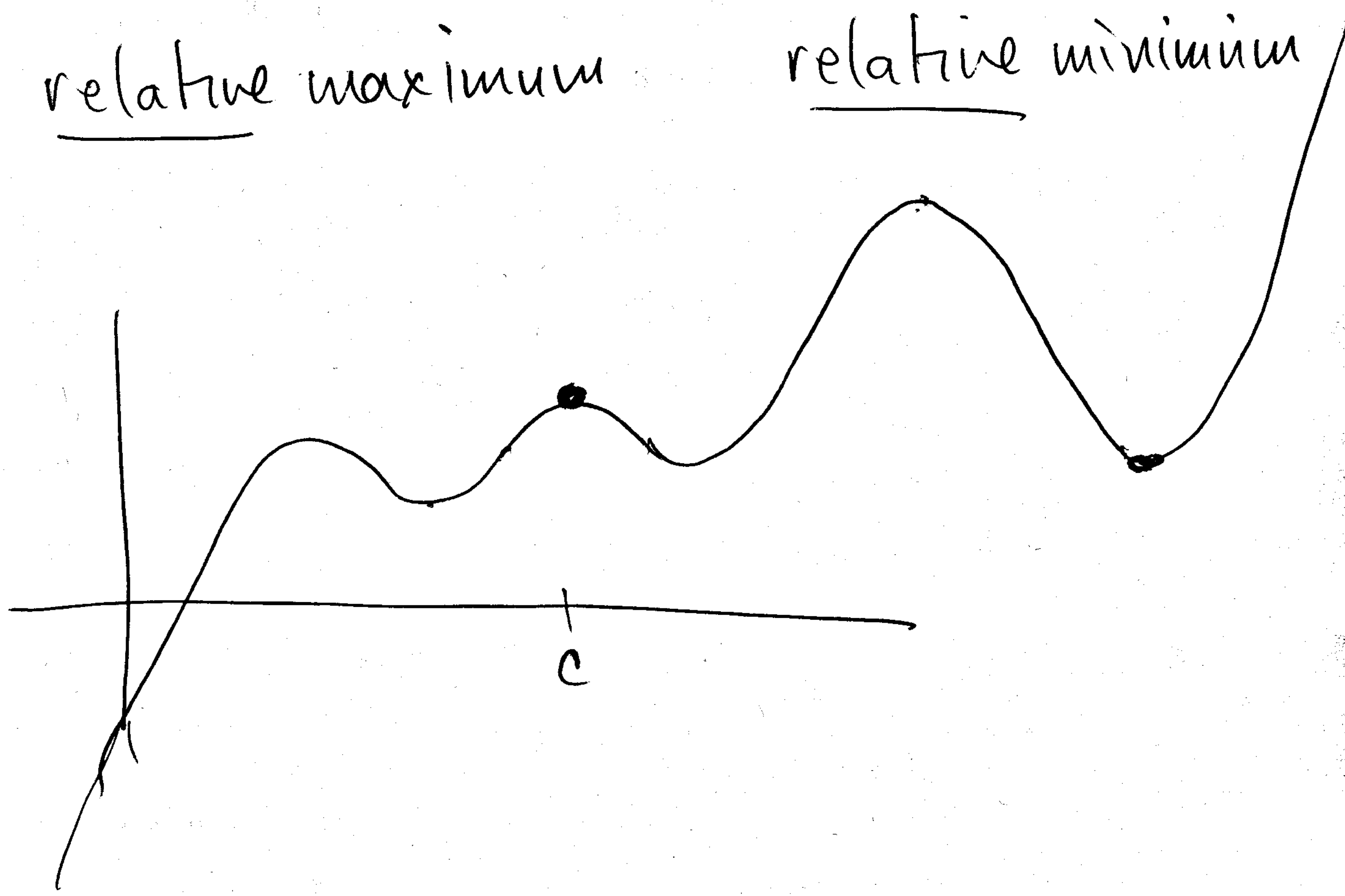
A number c in the domain of $f(x)$ is called a critical number if either $f'(c) = 0$ or $f'(c)$ does not exist. The corresponding point $(c, f(c))$ on the graph of $f(x)$ is called a critical point for $f(x)$.



relative maximum



relative minimum



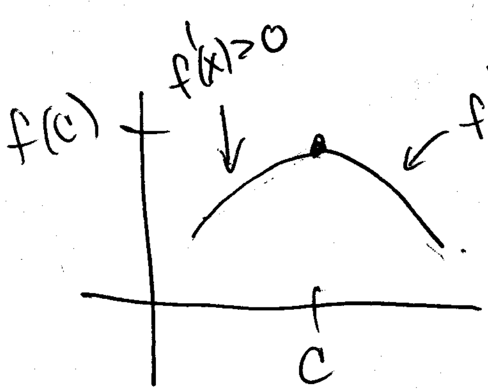
Relative Extrema

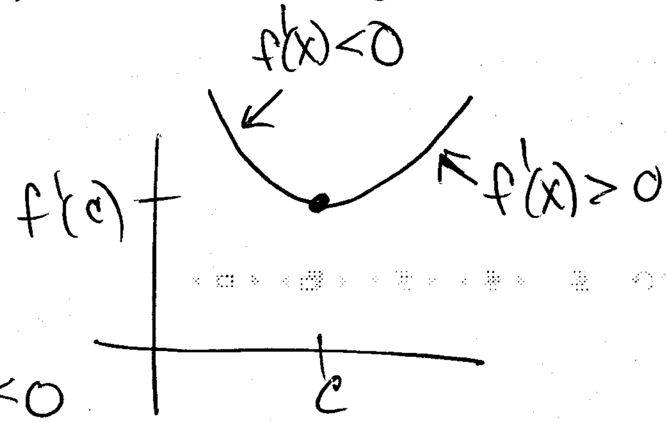
Relative extrema can only occur at critical points.

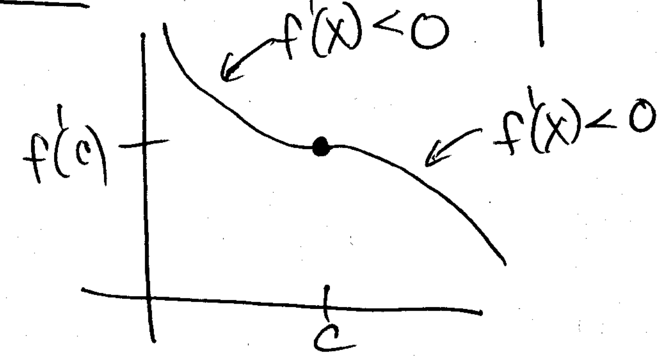
The First Derivative Test for Relative Extrema

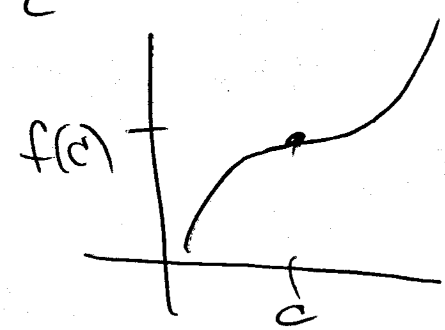
Let c be a critical number for $f(x)$. Then the critical point $(c, f(c))$ is

- A relative maximum if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c.
- A relative minimum if $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c.
- Not a relative extremum if $f'(x)$ has the same sign on both sides of c.









Relative Extrema

Example

Find all critical numbers of the function

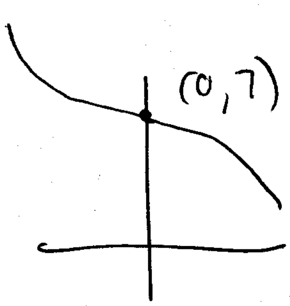
$$f(x) = 2x^5 - 5x^4 - 10x^3 + 7$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

Critical numbers: $x=0 \quad x=3 \quad x=-1$

Critical points: $(0,7) \quad (3,18) \quad (-1,10)$

$f(3) = 686 - 405 - 270 + 7 = 18$



(0,7) neither

(3,18) relative minimum

(-1,10) relative maximum

Relative Extrema

Example

Find all critical numbers of the function

$$F(x) = \frac{x^2}{x-3}$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

Critical numbers: $x=0 \quad x=3 \quad x=6$

Critical points: $(0,0) \quad (6,12)$

$(0,0)$ relative maximum

$(6,12)$ relative minimum

Relative Extrema

Example

Find all critical numbers of the function

$$f(x) = x\sqrt{4-x} = x(4-x)^{1/2}$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

$$f'(x) = x \cdot \frac{1}{2}(4-x)^{-1/2}(-1) + (4-x)^{1/2}$$

$$= -\frac{1}{2}x(4-x)^{-1/2} + (4-x)^{1/2}$$

$$= \frac{-x}{2(4-x)^{1/2}} + (4-x)^{1/2}$$

$$= \frac{-x}{2(4-x)^{1/2}} + \frac{2(4-x)^{1/2}(4-x)^{1/2}}{2(4-x)^{1/2}}$$

$$= \frac{-x+2(4-x)}{2(4-x)^{1/2}} = \frac{-3x+8}{2(4-x)^{1/2}}$$ //

Critical numbers: $x = \frac{8}{3}$ $x = 4$