1.5. Limits

- In this section we will learn how to evaluate and understand expressions of the form

$$\lim_{x \to c} f(x) = L$$

- The basic idea behind the notion of limit is this: We want to understand the behavior of a mathematical expression $f(x)$ *near* but not *at* the point $x = c$.
- The most practical use of this idea for this course is that limits can (sometimes) allow us to give meaning to mathematical expressions that evaluate to the meaningless form $0/0$.
- We will take three main approaches to understanding limits: (a) numerical, (b) graphical, and (c) algebraic. We will use the third approach most but the others should be understood.

Example: Numerical Approach. $\lim_{x \to 1} \frac{x^2-3x+2}{x-1} = -1$

Behavior of $f(x)$ for x near c

Consider the behavior of $f(x) = \dfrac{x^2 - 3x + 2}{x - 1}$ as x approaches 1.

x	0.8	0.9	0.99	1	1.01	1.1	1.2
$f(x)$	-1.2	-1.1	-1.01	*undefined*	-0.99	-0.9	-0.8

As x approaches 1, $f(x)$ approaches -1.

Definition

If $f(x)$ gets closer and closer to a number L as x gets closer and closer to c from both sides, then L is the *limit* of $f(x)$ as x approaches c. The behavior is expressed by

$$\lim_{x \to c} f(x) = L$$

39) $\lim_{x \to 2} (x^2 - x)$

$f(x) = x^2 - 2$

$f(2) = 2$ //

x	1.9	1.95	1.999	2	2.001	2.01	2.1
$f(x)$	1.71	1.8525	1.997001	X	2.003001	2.0301	2.31

$\lim_{x \to 2} (x^2 - x) = 2$

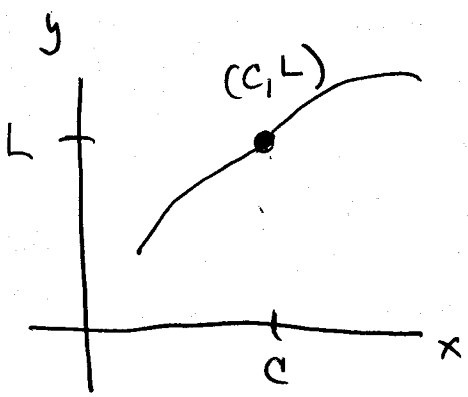
42) $\lim_{x \to -1} \frac{x^3+1}{x+1} = 3$

x	-1.1	-1.01	-1.005	-1	-.99	-.9	-.8
$f(x)$	3.31	3.0301	3.015025	X	2.9701	2.71	2.44

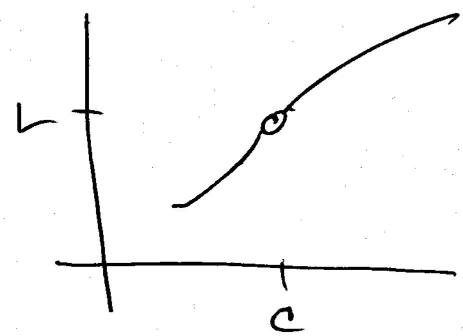
Example: Graphical Approach.

It is important to remember that limits describes the behavior of a function *near* a particular point, not necessarily *at* the point itself. Note that the limit in each case is the same and is independent of the value of the function at $x = c$ or even if the function is defined at $x = c$.

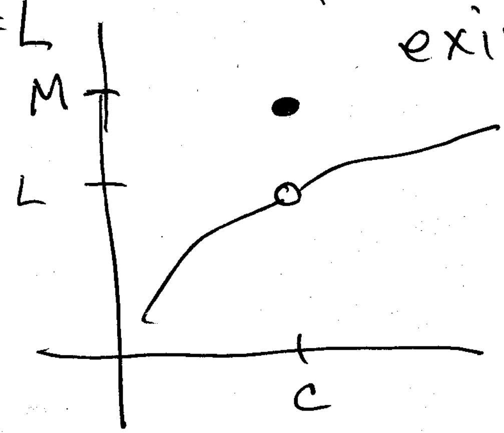
Three functions for which $\lim_{x \to c} f(x) = L$



$f(c) = L$



$f(c)$ does not exist

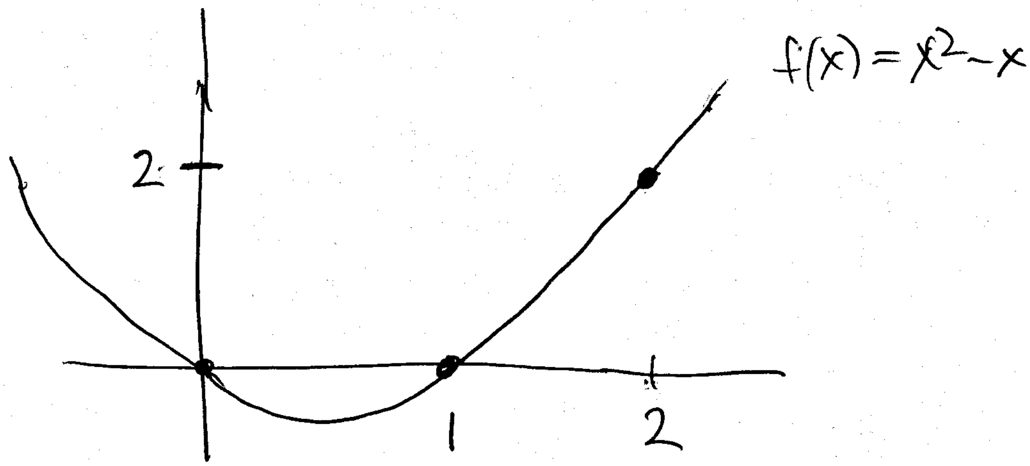


$f(c) = M$

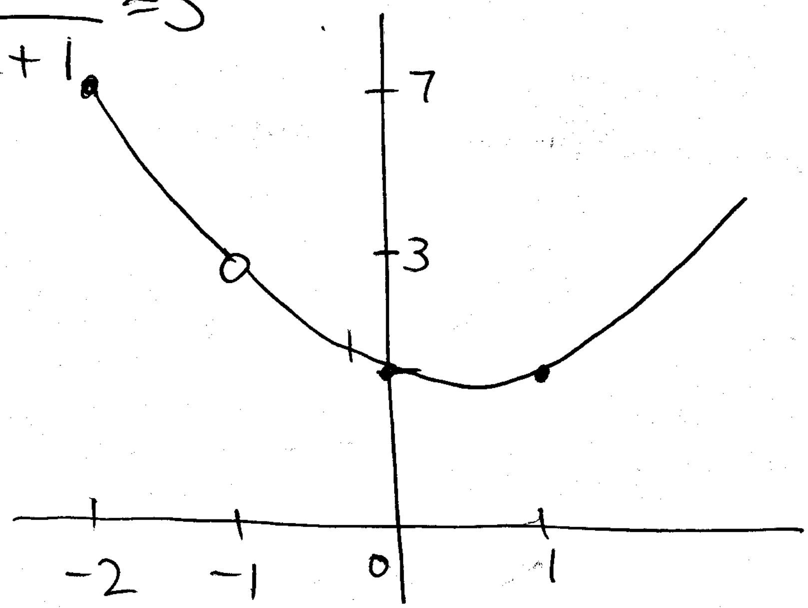
$\cancel{\lim_{x \to c} f(x) = M}$

$f(c) \neq L$

e.g. $\lim_{x \to 2} (x^2 - x) = 2$



$\lim_{x \to -1} \frac{x^3+1}{x+1} = 3$



Example: Algebraic Approach.

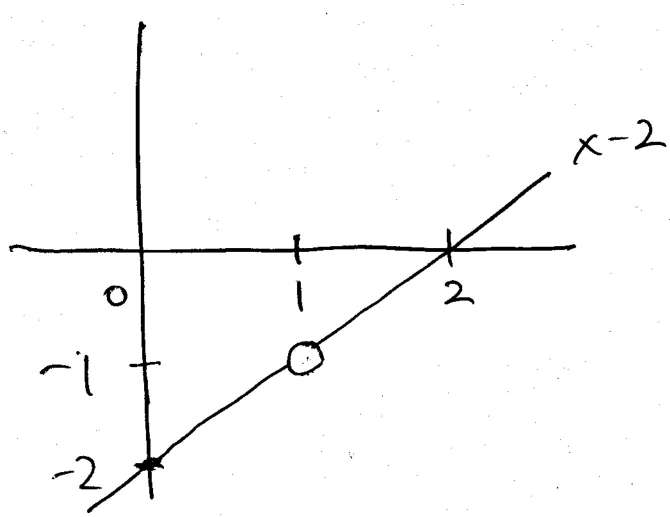
We are looking at $\lim_{x\to 1} \dfrac{x^2-3x+2}{x-1} = -1$

- Note that as long as $x \neq 1$,

$$\frac{x^2-3x+2}{x-1} = \frac{\cancel{(x-1)}(x-2)}{\cancel{x-1}} = (x-2)$$

. Of course, if $x = 1$ then $\dfrac{x^2-3x+2}{x-1}$ is undefined.

- Therefore, $\lim_{x\to 1} \dfrac{x^2-3x+2}{x-1} = \lim_{x\to 1}(x-2) = 1-2 = -1$ as we shall see.



$$\lim_{x \to -1} \frac{x^3+1}{x+1} = \lim_{x \to -1} \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{x+1}} = \lim_{x \to -1} (x^2-x+1)$$

$$x^3+1 = (x+1)(x^2-x+1)$$

$$= (-1)^2 - (-1) + 1$$

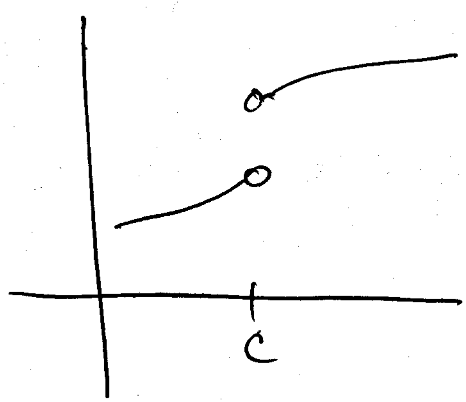
$$= 1+1+1 = 3$$

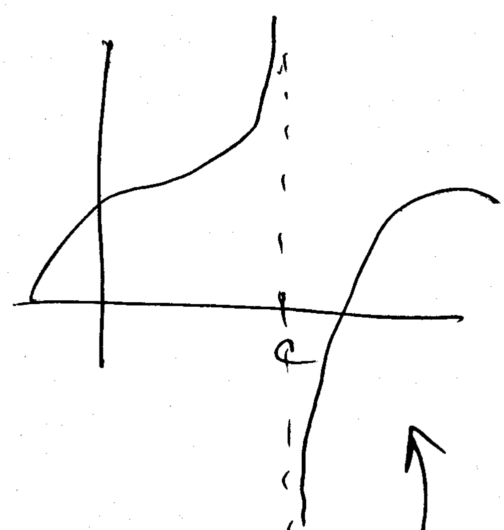
$$\lim_{x \to 1} \frac{x^3+1}{x+1} = 1$$

$$\begin{array}{r|l}
 & x^2 - x + 1 \\
x+1 & x^3+0x^2+0x+1 \\
 & -(x^3+x^2) \\
\hline
 & -x^2+0x \\
 & -(-x^2-x) \\
\hline
 & x+1 \\
 & x+1 \\
\hline
 & 0
\end{array}$$

Functions for which the limit does not exist

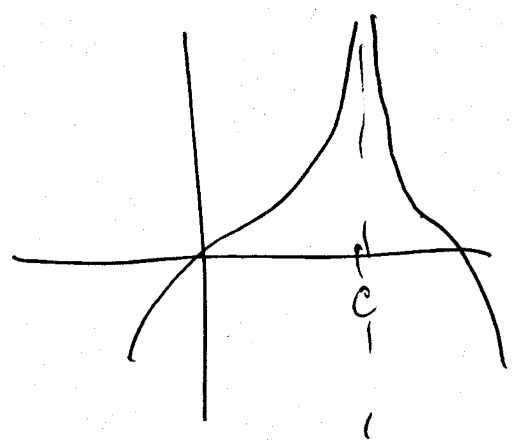
It is possible that the limit $\lim_{x \to c} f(x)$ does not exist.





$\lim_{x \to c} f(x)$ does not exist.

$f(x)$ becomes infinite as $x \to c$.
(vertical asymptote)



Properties of Limits

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, then

$$\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

$$\lim_{x \to c}[f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$$

$$\lim_{x \to c}[kf(x)] = k \lim_{x \to c} f(x)$$

$$\lim_{x \to c}[f(x)g(x)] = [\lim_{x \to c} f(x)][\lim_{x \to c} g(x)]$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \quad \text{if } \lim_{x \to c} g(x) \neq 0$$

$$\lim_{x \to c}[f(x)]^p = [\lim_{x \to c} f(x)]^p \quad \text{if } [\lim_{x \to c} f(x)]^p \text{ exists}$$

Computation of Limits

Limits of Polynomials and Rational functions

If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \to c} p(x) = p(c) \quad \text{and} \quad \lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \quad \text{if } q(c) \neq 0$$

Example

Find $\lim_{x \to 2} (x^2 - 4x + 7)$. $= (2)^2 - 4(2) + 7 = 4 - 8 + 7 = 3$ //

Example

Find $\lim_{x \to 1} \frac{x+3}{2x+1}$. $= \frac{1+3}{2(1)+1} = \frac{4}{3}$ //

$\frac{6}{-.5} = -12$

$1.9 - 2 = -.1$

$\frac{6.8}{-.1} = -68$

$\frac{8}{.5} = 16$ $\quad$ $\frac{7.2}{.1} = 72$

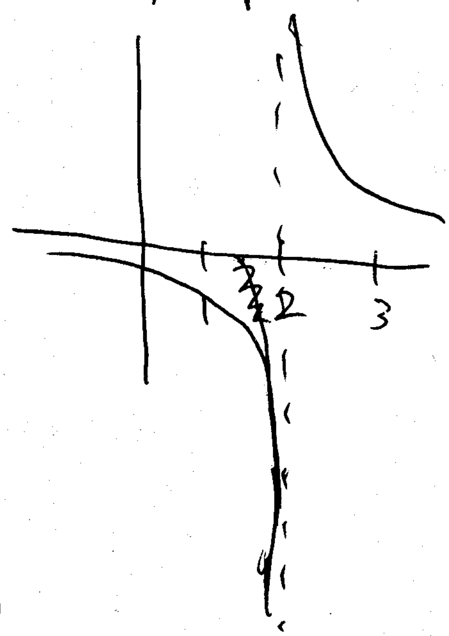
Computation of Limits

Evaluates to $\frac{7}{0}$. I have a vertical asymptote.

Example

Find $\lim\limits_{x \to 2} \frac{2x+3}{x-2}$. DOES NOT EXIST

x	1.5	1.9	2.1	2.5		
$\frac{2x+3}{x-2}$	-12	-68	72	16		



Example Evaluates to $\frac{0}{0}$

Find $\lim\limits_{x \to 2} \frac{x^2+x-6}{x-2}$.

Factor. $x^2+x-6 = (x-2)(x+3)$

$$\lim_{x \to 2} \frac{x^2+x-6}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+3)}{x-2} = \lim_{x \to 2} x+3 = 5$$

Limits involving Infinity

Limits at infinity

If the values of $f(x)$ approach the number L as x gets larger and larger,

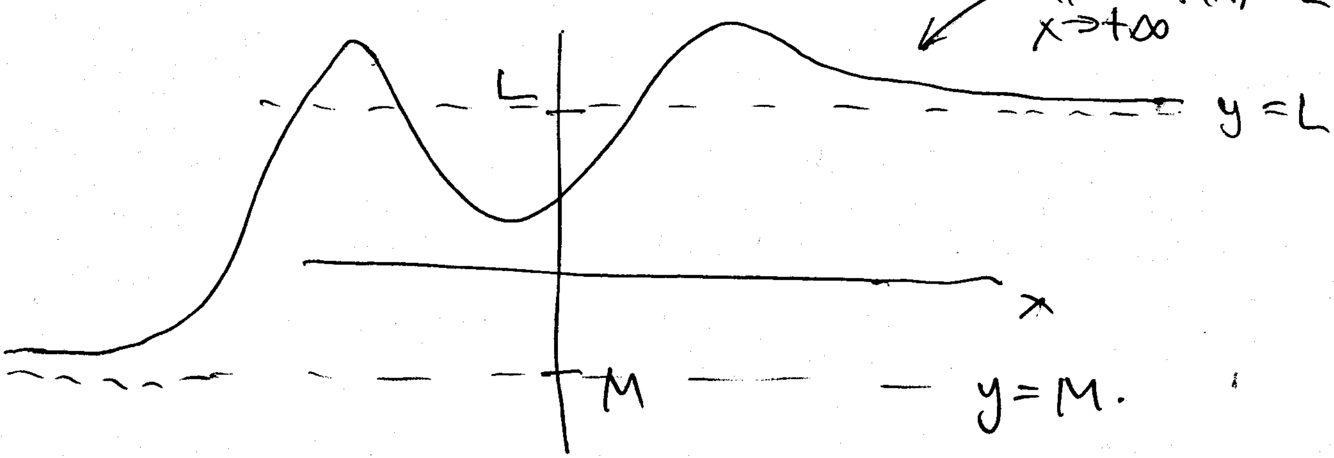
$$\lim_{x\to+\infty} f(x) = L$$

If the values of $f(x)$ approach the number L as x gets larger and larger negatively,

$$\lim_{x\to-\infty} f(x) = M.$$

Graphically, $\lim_{x\to\pm\infty} f(x) = L$ means that $f(x)$ has a *horizontal asymptote* at the line $y = L$.

$\lim_{x\to+\infty} f(x) = L$



Limits involving Infinity

Example

If $k > 0$ and x^k is defined for all x, then for any constant A,

$$\lim_{x \to \pm\infty} \frac{A}{x^k} = 0.$$

Example

Find $\lim\limits_{x \to +\infty} \dfrac{1 - 2x^3}{2x^3 - 5x + 4}$. $= \lim\limits_{x \to +\infty} \dfrac{(1-2x^3)}{2x^3-5x+4} \dfrac{\frac{1}{x^3}}{\frac{1}{x^3}}$

$$= \lim_{x \to +\infty} \frac{\frac{1}{x^3} - 2}{2 - \frac{5}{x^2} + \frac{4}{x^3}} = \frac{0-2}{2-0+0} = -1 \;//$$

Example

Find $\lim\limits_{x \to -\infty} \dfrac{x^2 + 2x - 3}{1 - 3x - x^3}$.

$$= \lim_{x \to -\infty} \frac{x^2+2x-3}{1-3x-x^3} \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)} = \lim_{x \to -\infty} \frac{\frac{1}{x} + \frac{2}{x^2} - \frac{3}{x^3}}{\frac{1}{x^3} - \frac{3}{x^2} - 1} = \frac{0+0-0}{0-0-1} = 0 \;//$$

Limits involving Infinity

Infinite Limits

If $f(x)$ increases without bound as $x \to c$, we write

$$\lim_{x \to c} f(x) = +\infty.$$

If $f(x)$ decreases without bound as $x \to c$, then

$$\lim_{x \to c} f(x) = -\infty.$$

Graphically, $\lim_{x \to c} f(x) = \pm\infty$ means that $f(x)$ has a *vertical asymptote* at the line $x = c$.

Example

Find $\displaystyle \lim_{x \to -1/2} \frac{1 - 3x^3}{2x + 1}$.