1.4. Functional Models

Basic Goal : Developing mathematical methods for dealing with practical problems.

Mathematical Modeling

Stage. 1 *(Formulation)* Identify key variables and establish equations relating those variables.

Stage. 2 *(Analysis of the Model)* Use mathematical methods to analyze or "solve" the model.

Stage. 3 *(Interpretation)* Any conclusions from the analysis are applied to the original real world problem.

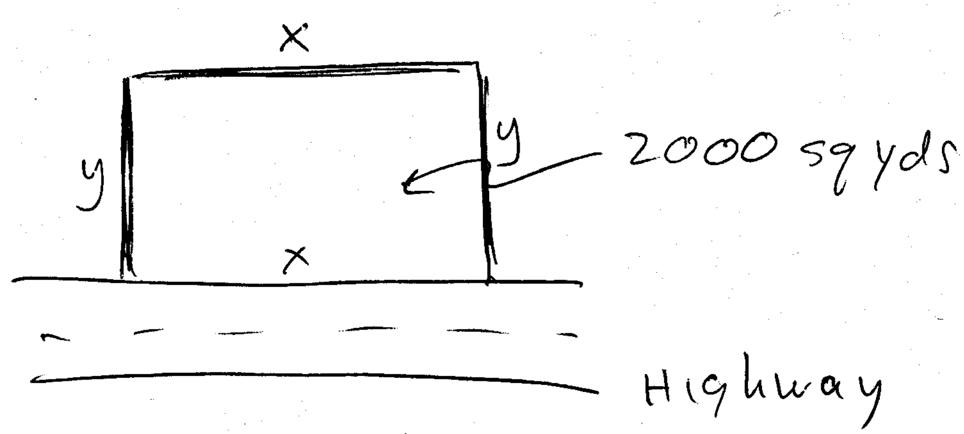
Stage. 4 *(Testing and Adjustment)* The model is tested by gathering new data.

Elimination of variables

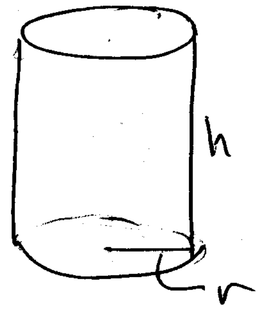
Example

The highway department is planning to build a picnic area along a major highway. It is to be rectangular with an area of 2000 square yards and is to be fenced off on the three sides not adjacent to the highway. Express the number of yards of fencing required as a function of the length of the unfenced side.

1. Draw a picture



Example



h = height of can (inches)

r = radius of can (inches)

b = area of base $= \pi r^2$ (sq inches)

C = cost of construction (cents)

$C = 3$ (area of top + area of ~~bottom~~ base)

$+2$ (area of side)

$$= 3(2\pi r^2) + 2(2\pi r h)$$

$$= 6\pi r^2 + 4\pi r h$$

Eliminate h.

$$24\cancel{\pi} = \cancel{\pi} r^2 h$$

$$h = \frac{24}{r^2}$$

$$C = 6\pi r^2 + \frac{4\pi \cancel{r} \cdot 24}{r^{\cancel{2}}} = 6\pi r^2 + \frac{96\pi}{r}$$

$$F = x + \frac{4000}{x}$$

$$C = 6\pi r^2 + \frac{96\pi}{r}$$

$F(1) = 4001$

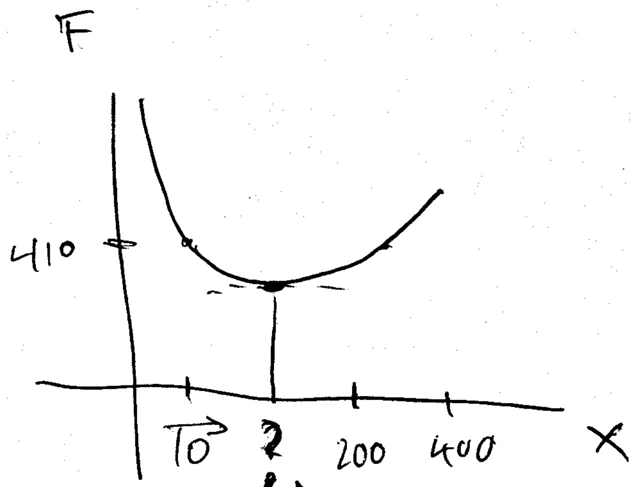
$F(2) = 2002$

$F(10) = 410$

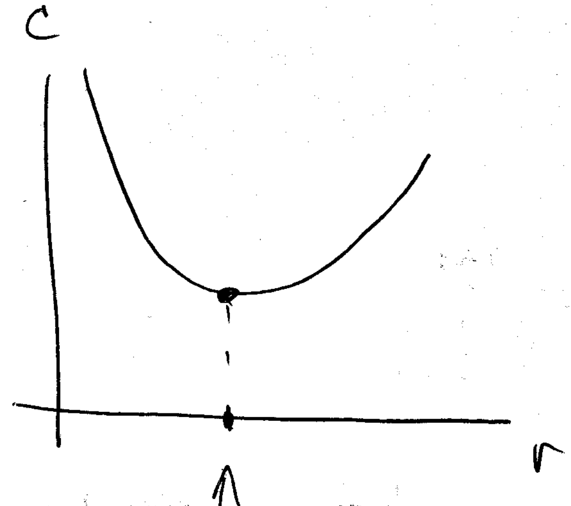
$F(100) = 140$

$F(200) = 220$

$F(400) = 410$



x that gives least amount of fencing



$C(1) = 102\pi$

$C(2) = 72\pi$

$C(3) = 86\pi$

$C(4) = 120\pi$

radius giving minimum cost.

Example 1.4.5

x = price per ream (dollars)

n = # sold.

P = profit (dollars)

a. $P = (\#\text{reams sold})(x) - (2)(\#\text{reams sold})$

$= \underbrace{(\#\text{reams sold})}_{n}(x-2)$

Find n in terms of x.

$x=5$	$x=6$	$x=7$	
$n=4000$	$n=3600$	$n=3200$	...

linear relationship slope: -400

point on line $(5, 4000)$

$y - 4000 = -400(x-5)$ (P-S formula)

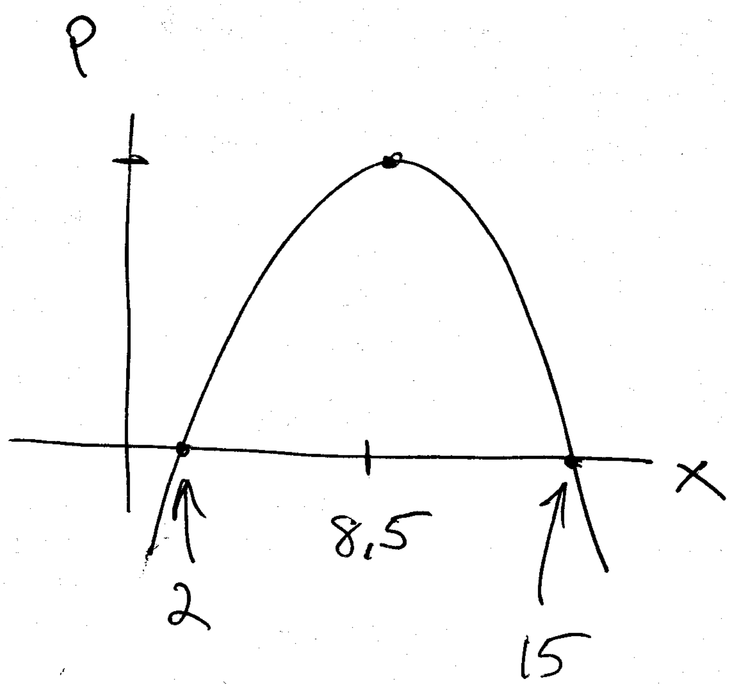
$n = 4000 - 400(x-5)$

$= 4000 - 400x + 2000$

$= 6000 - 400x$

$P = (6000 - 400x)(x-2)$ //

b. $P = -400x^2 + 6800x - 12000$



$x = \$8,50$ per ream maximizes profit

Vertex: $x = \frac{-6800}{-800} = 8,5$

Modeling in Business and Economics

Example

A closed box with a square base is to have a volume of 250 cubic meters. The material for the top and bottom of the box costs \$2 per square meter, and the material for the sides costs \$1 per square meter. Express the construction cost of the box as a function of the length of its base.

Market Equilibrium

Demand function $D(x)$: the unit price at which all x units are demanded (sold) in the marketplace.
Supply function $S(x)$: the unit price at which producers are willing to supply x units to the marketplace.

The Law of Supply and Demand

- In a competitive market environment, supply tends to equal demand.
- When this occurs, the market is said to be in *equilibrium*.
- The corresponding unit price is called the *equilibrium price*.
- The market has a *shortage* when demand exceeds supply.
- The market has a *surplus* when supply exceeds demand.

Market Equilibrium

Example

Supply function and demand function for a particular commodity are given as follows:

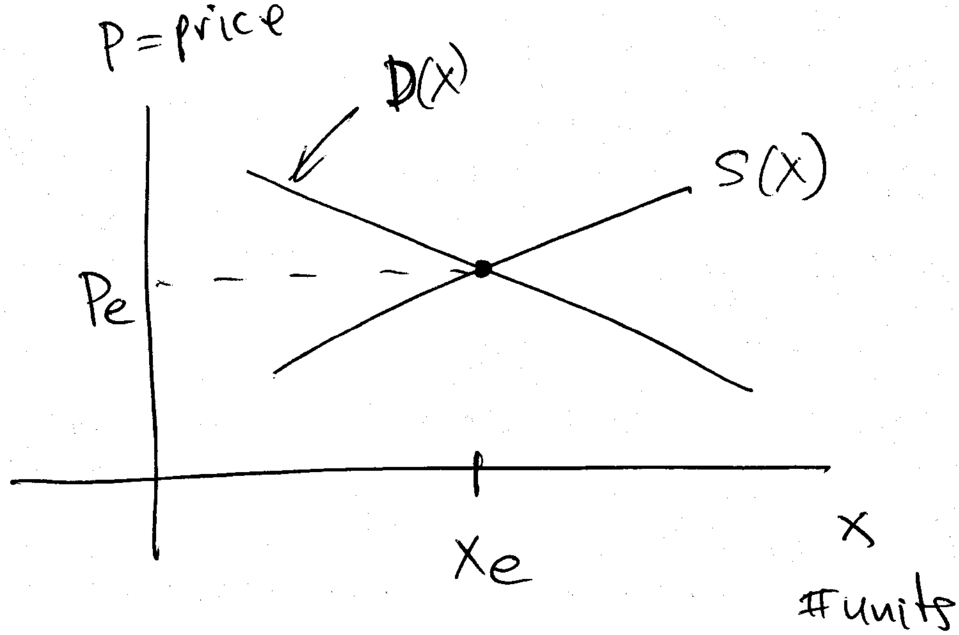
$$S(x) = 4x + 200 \text{ and } D(x) = -3x + 480$$

(a) Find the value of x_e for which equilibrium occurs and the corresponding equilibrium price p_e.

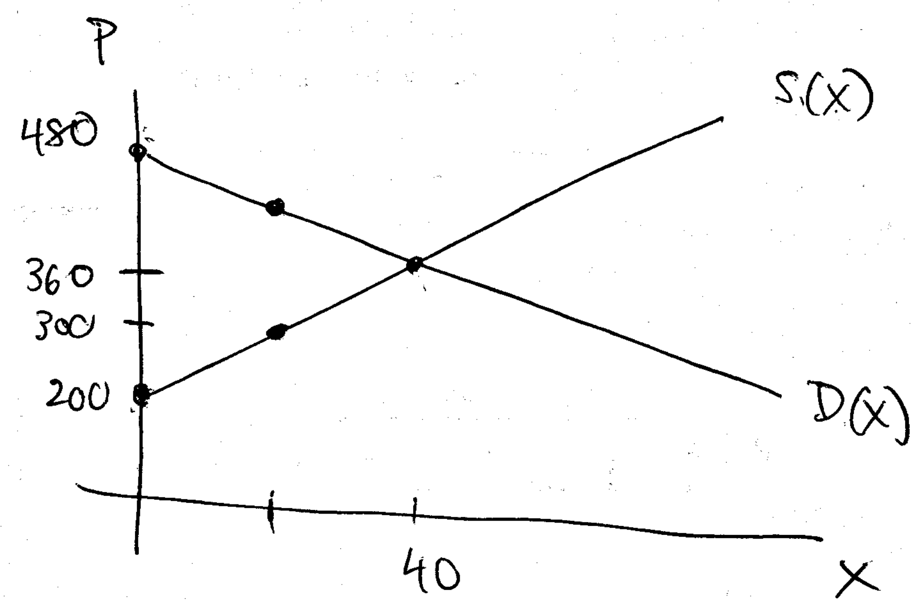
(b) Sketch the graphs of the supply and demand curves.

(c) For what values of x is there a market shortage? A market surplus?

Market Equilibrium



Example



(a) $4x + 200 = -3x + 480$

$7x = 280$

$x_e = 40$ units

$P_e = 360$ dollars.

Break-even analysis

The point at which the total revenue curve and the total cost curve cross is called the *break-even point*.

Example

A furniture manufacturer can sell dining room tables for \$70 a piece. The manufacturer's total cost consists of a fixed overhead of \$8,000 plus production costs of \$30 per table.

(a) How many tables must the manufacturer sell to break even?

(b) How many tables must the manufacturer sell to make a profit of \$6,000?

(c) What will be the manufacturer's profit or loss if 150 tables are sold?

(d) Graph the manufacturer's total revenue and total cost function. Explain how the overhead can be read from the graph.

2. Identify variables

x - length of fence (yds)

y - width of fence (yds)

F - length of fencing required. (yds)

Express F in terms of x.

3. $F = x + 2y$

Eliminate y by expressing it as a function of x.

$$xy = 2000 \qquad y = \frac{2000}{x}$$

$$F = x + \frac{4000}{x}$$