Approximation by Increments

If $f(x)$ is differentiable at $x = x_0$ and Δx is a small change in x, then

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

or, equivalently, if $\Delta f = f(x_0 + \Delta x) - f(x_0)$, then

$$\Delta f \approx f'(x)\Delta x$$

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

$$f'(x_0) \approx \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \qquad \text{if } \Delta x \text{ small}$$

$$\Delta x f'(x_0) \approx f(x_0+\Delta x) - f(x_0) = \Delta f$$

$$f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

eg

#8) $f(x) = \dfrac{x}{x+1} - 3$

Estimate Δf as x changes from 4 to 3.8

$x+\Delta x$ $\quad$ x

3.8 $\quad$ 4

$\therefore \Delta x = -.2$

$\cancel{\Delta f \approx f(4) - f(3.8) = f(x+\Delta x) - f(x)}$

$\Delta f = f(3.8) - f(4) = f(x+\Delta x) - f(x)$

$\Delta f \approx f'(x)\Delta x = f'(4)(-.2)$

$f'(x) = \dfrac{(x+1)(1) - (x)(1)}{(x+1)^2} = \dfrac{\cancel{x}+1-\cancel{x}}{(x+1)^2} = \dfrac{1}{(x+1)^2}$

$f'(4) = \dfrac{1}{(4+1)^2} = \dfrac{1}{25}$

$\Delta f \approx \left(\frac{1}{25}\right)(-.2) = (.04)(-.2) = -.008$

$\Delta f = f(3.8) - f(4) = \left(\dfrac{3.8}{4.8} \cancel{-3}\right) - \left(\dfrac{4}{5} \cancel{-3}\right)$

$f(3.8) = \dfrac{3.8}{3.8+1} - 3 = \dfrac{3.8}{4.8} - 3$

$\cong .791667 - .8$

$= -.008333\ldots$

Approximation by Increments

Example

A 5-year projection of population trends suggests that t years from now, the population of certain community will be $P(t) = -t^3 + 9t^2 + 48t + 200$ thousand.

(a) Find the rate of change of population $R(t) = P'(t)$ with respect to time t.

(b) At what rate does the population growth rate $R(t)$ change with respect to time?

(c) Use increments to estimate how much ~~$R(t)$~~ $P(t)$ changes during the first month of the fourth year. What is the actual change in ~~$R(t)$~~ $P(t)$ during this time period?

(a) $R(t) = P'(t) = -3t^2 + 18t + 48 \ \frac{\text{thousands}}{\text{year}}$

(c) $P'(4) = -3(4)^2 + 18(4) + 48$

$= -48 + 72 + 48 = 72 \ \frac{\text{thousands}}{\text{year}}$

$\Delta P \approx P'(4)\underbrace{\left(\frac{1}{12}\right)}_{\Delta t} = (72)\left(\frac{1}{12}\right) = 6$ thousand

Exact change: $P(4+\frac{1}{12}) - P(4)$

$= P(4.0833\cdots) - P(4)$

$\approx 478 - 472 = 6$

~~477.97~~ Estimate was very good.

Approximation by Increments

Approximation formula for Percentage Change

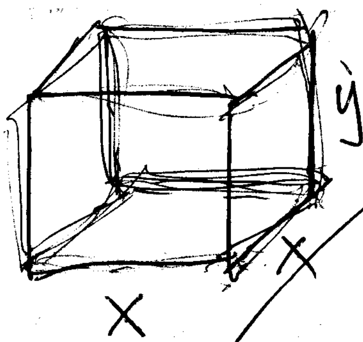
If Δx is a (small) change in x, the corresponding percentage change in the function $f(x)$ is

$$\text{Percentage change in } f = 100\frac{\Delta f}{f(x)} \approx 100\frac{f'(x)\Delta x}{f(x)}$$

Example

Use increments to estimate the percentage change in the function $f(x) = 3x + \dfrac{2}{x}$ as x decreases from 5 to 4.6.

10)



$$1500 = x^2 y$$

$$S = x^2 + xy + xy + xy$$
$$+ xy + x^2$$
$$= 2x^2 + 4xy$$

$$y = \frac{1500}{x^2}$$

$$S = 2x^2 + 4\cancel{x}\left(\frac{1500}{x^{\cancel{2}}}\right)$$
$$= 2x^2 + \frac{6000}{x}$$

#4) x – one number

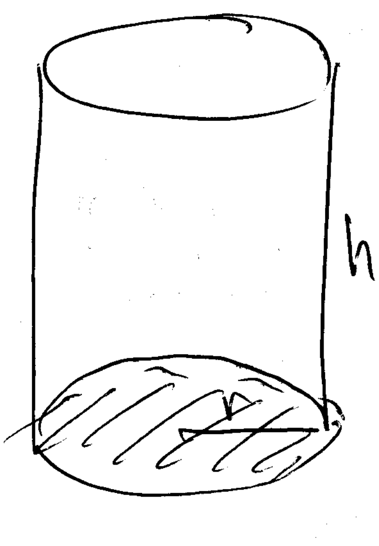
y – the other

$x + y = 18 \longrightarrow y = 18 - x$

p – product of #s

$p = xy$

$p = x(18 - x)$ ✓



$S = 2\pi r h$

$27\pi = 2\pi r h + \pi r^2$

$27\cancel{\pi} - \cancel{\pi} r^2 = 2\cancel{\pi} r h$

$h = \dfrac{27 - r^2}{r}$

$V = \pi r^2 h = \pi r^{\cancel{2}} \left(\dfrac{27 - r^2}{\cancel{r}} \right) = \pi r (27 - r^2)$