5.3. The Definite Integral and the Fundamental Theorem of Calculus

Area under a Curve

Let $f(x)$ be continuous and $f(x) \geq 0$ on the interval $a \leq x \leq b$. Then the region under the curve $y = f(x)$ over the interval $a \leq x \leq b$ has area

$$A = \lim_{n \to +\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

where x_j is the left endpoint of the jth subinterval if the interval $a \leq x \leq b$ is divided into n equal parts, each of length $\Delta x = \dfrac{b-a}{n}$.

The Definite Integral

Let $f(x)$ be a continuous function on $a \leq x \leq b$. Subdivide the interval $a \leq x \leq b$ in n equal parts, each of width $\Delta x = \dfrac{b-a}{n}$, and choose a number x_k from the kth subinterval. Form the sum

$$[f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

called the Riemann Sum.
Then the definite integral of f on the interval $a \leq x \leq b$, denoted by $\int_a^b f(x)\,dx$, is the limit of the Riemann sum as $n \to +\infty$; that is,

$$\int_a^b f(x)\,dx = \lim_{n \to +\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

The Definite Integral

The Fundamental Theorem of Calculus

If the function $f(x)$ is continuous on the interval $a \le x \le b$, then

$$\int_a^b f(x)\ dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$ on $a \le x \le b$.

Example (#4)

Evaluate $\displaystyle\int_1^4 (5 - 2t)\ dt$.

The Definite Integral

Area as a Definite Integral

If $f(x)$ is continuous and $f \geq 0$ on the interval $a \leq x \leq b$, then the region under the curve $y = f(x)$ over the interval $a \leq x \leq b$ has area given by the definite integral $\int_a^b f(x)\ dx$.

Example (#38)

Find the area of the region that lies under the curve $y = \sqrt{x}(x + 1)$ over the interval $0 \leq x \leq 4$.

Rules of Definite Integrals

Let f and g be continuous on $a \leq x \leq b$. Then

- The constant multiple rule:

$$\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx \quad \text{for constant } k$$

- The sum rule:

$$\int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

- The difference rule:

$$\int_a^b [f(x) - g(x)]\ dx = \int_a^b f(x)\ dx - \int_a^b g(x)\ dx$$

Rules of Definite Integrals

Let f and g be continuous on $a \leq x \leq b$. Then

- $\int_a^a f(x)\ dx = 0$
- $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$
- The subdivision rule:

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

Rules of Definite Integrals

Example (#36)

Let $f(x)$ and $g(x)$ be continuous on $-3 \leq x \leq 1$ and satisfy

$$\int_{-3}^{1} f(x)\ dx = 0 \quad \int_{-3}^{1} g(x)\ dx = 4$$

Evaluate $\int_{-3}^{1} [2f(x) + 3g(x)]\ dx$.

Rules of Definite Integrals

Example (#32)

Let $g(x)$ be continuous on $-3 \leq x \leq 2$ and satisfies

$$\int_{-3}^{2} g(x)\ dx = -2 \quad \int_{-3}^{1} g(x)\ dx = 4$$

Evaluate $\int_{1}^{2} g(x)\ dx$.