

3.1. Increasing and Decreasing Functions; Relative Extrema

Increasing and Decreasing Functions

Let $f(x)$ be a function defined on the interval $a < x < b$, and let x_1 and x_2 be two numbers in the interval. Then

- ▶ $f(x)$ is **increasing** on the interval if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.
- ▶ $f(x)$ is **decreasing** on the interval if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$.

Intervals of Increase and Decrease

Procedure for using the derivative to determine intervals of increase and decrease

- Step 1.** Find all values of x for which $f'(x) = 0$ or $f'(x)$ is not continuous, and mark these numbers on a number line. This divides the line into a number of open intervals.
- Step 2.** Choose a test number c from each interval $a < x < b$ determined in Step 1 and evaluate $f'(c)$. Then
- ▶ If $f'(c) > 0$, $f(x)$ is **increasing** on $a < x < b$.
 - ▶ If $f'(c) < 0$, $f(x)$ is **decreasing** on $a < x < b$.

Intervals of Increase and Decrease

Example

Find the intervals of increase and decrease for the function

$$f(x) = 2x^5 - 5x^4 - 10x^3 + 7$$

Intervals of Increase and Decrease

Example

Find the intervals of increase and decrease for the function

$$F(x) = \frac{x^2}{x - 3}.$$

Relative Extrema

Definition

- ▶ The graph of the function $f(x)$ is said to have a **relative maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in an interval $a < x < b$ containing c .
- ▶ Similarly, the graph has a **relative minimum** at $x = c$ if $f(c) \leq f(x)$ on such an interval.
- ▶ Collectively, the relative maxima and minima of f are called its **relative extrema**.

Definition

A number c in the domain of $f(x)$ is called a **critical number** if either $f'(c) = 0$ or $f'(c)$ does not exist. The corresponding point $(c, f(c))$ on the graph of $f(x)$ is called a **critical point** for $f(x)$.

Relative Extrema

Relative extrema can only occur at critical points.

The First Derivative Test for Relative Extrema

Let c be a critical number for $f(x)$. Then the critical point $(c, f(c))$ is

- ▶ A **relative maximum** if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c .
- ▶ A **relative minimum** if $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c .
- ▶ **Not a relative extremum** if $f'(x)$ has the same sign on both sides of c .

Relative Extrema

Example

Find all critical numbers of the function

$$f(x) = 2x^5 - 5x^4 - 10x^3 + 7$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

Relative Extrema

Example

Find all critical numbers of the function

$$F(x) = \frac{x^2}{x - 3}$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

Relative Extrema

Example

Find all critical numbers of the function

$$f(x) = x\sqrt{4-x}$$

and classify each critical point as a relative maximum, a relative minimum, or neither.

Sketching the graph

Prodecure for sketching the graph of a continuous function using the derivative

- Step 1.** Determine the domain of $f(x)$. Set up a number line restricted to include only those numbers in the domain.
- Step 2.** Find $f'(x)$ and mark each critical number on the restricted number line. Then analyze the sign of $f'(x)$ to determine the intervals of increase and decrease for $f(x)$.
- Step 3.** For each critical number c , find $f(c)$ and plot the critical point $P(c, f(c))$ on a plane. Plot intercepts and other key points that can be easily found.
- Step 4.** Sketch the graph of f as a smooth curve joining the critical points in such a way that it rises where $f'(x) > 0$, falls where $f'(x) < 0$, and has a horizontal tangent where $f'(x) = 0$.

Sketching the graph

Example

Use calculus to sketch the graph of

$$f(x) = 2x^5 - 5x^4 - 10x^3 + 7$$

Sketching the graph

Example

Use calculus to sketch the graph of

$$F(x) = \frac{x^2}{x - 3}.$$

Sketching the graph

Example

Use calculus to sketch the graph of

$$f(x) = \frac{x + 1}{x^2 + x + 1}.$$