2.4. The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is in turn a differentiable function of x, then the composite function $f(g(x))$ is a differentiable function of x whose derivative is given by the product

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

or, equivalently, by

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

The Chain Rule

Example

Compute the derivative $\frac{dy}{dx}$ and simplify the answer if

$$y = u^2 - 3u + 4; \quad u = 1 - x^2$$

The Chain Rule

Example

Compute the derivative $\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}}$ if

$$y = u^2 - 2u + 2; \quad u = \frac{1}{x}$$

The Chain Rule

Sometimes when dealing with a composite function $y = f(g(x))$ it may help to think of f as the "outer" function and g as the "inner" function. Then the chain rule says that the derivative of $y = f(g(x))$ with respect to x is given by *the derivative of the outer function evaluated at the inner function times the derivative of the inner function*.

Example

Differentiate the following function and simplify the answer.

$$h(x) = \sqrt{x^6 - 3x^2}$$

The General Power Rule

For any real number n and differentiable function h,

$$\frac{d}{dx}[h(x)]^n = n[h(x)]^{n-1}\frac{d}{dx}[h(x)]$$

Example

Differentiate the following function and simplify the answer.

$$f(x) = (t^4 - 4t^2 + 4)^6$$

Combination with other rules

Example

Differentiate the following function and simplify the answer.

$$f(x) = (2x + 1)^4(3x - 5)^2$$

Combination with other rules

Example

Differentiate the following function and simplify the answer.

$$F(x) = \frac{(1-2x)^3}{(3x+1)^2}$$

Higher derivatives

Example

Find the second derivative of the given function

$$y = (1 - x^2)^3$$