2.1 The Derivative

The derivative of a function

The *derivative* of the function $f(x)$ with respect to x is the function $f'(x)$ given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The process of computing the derivative is called *differentiation*, and we say that $f(x)$ is *differentiable* at $x = c$ if $f'(c)$ exists.

Example

Find the derivative of the function $f(x) = x^2 - 2x$.

Slope as a Derivative

The slope of the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$ is $m_{tan} = f'(c)$.

Example

Find the equation of the tangent line to the curve $y = x^2 - 2x$ at the point where $x = -1$.

Instantaneous Rate of Change as a Derivative

The rate of change of $f(x)$ with respect to x when $x = c$ is given by $f'(c)$.

Example

A toy rocket rises vertically in such a way that t seconds after lift-off, it is

$$h(t) = -\frac{1}{2}t^2 + 20t$$

feet above ground.

a. What is the (instantaneous) velocity of the rocket at lift-off?

b. What is its velocity after 10 seconds?

Significance of the sign of $f'(x)$

If the function f is differentiable at $x = c$, then

$$f \text{ is } \textit{increasing} \text{ at } x = c \text{ if } f'(c) > 0$$

and

$$f \text{ is } \textit{decreasing} \text{ at } x = c \text{ if } f'(c) < 0$$

Example

c. At lift-off, is the rocket rising?

d. Is the rocket rising after 30 seconds?

Derivative Notation

The derivative $f'(x)$ of $y = f(x)$ is sometimes written as

$$\frac{dy}{dx} \text{ or } \frac{df}{dx}$$

In this notation, $f'(c)$ is written as

$$\left.\frac{dy}{dx}\right|_{x=c} \text{ or } \left.\frac{df}{dx}\right|_{x=c}$$

Example

Find the rate of change $\frac{dy}{dx}$ of $y = 5 - x^2$ at the point where $x = 2$.

Differentiability and Continuity

Continuity of a differentiable function

If the function $f(x)$ is differentiable at $x = c$, then it is also continuous at $x = c$. This means that for $f(x)$ to be differentiable at $x = c$ it must at least be continuous there, but *more is required*. There are functions that are continuous at a point but not differentiable there.

Examples of nondifferentiability

Each of the functions below is continuous at $x = 0$ but not differentiable at $x = 0$.

- Vertical tangent: $f(x) = x^{1/3}$
- Cusp: $f(x) = x^{2/3}$
- Corner: $f(x) = |x|$