1.5. Limits

- In this section we will learn how to evaluate and understand expressions of the form

$$\lim_{x \to c} f(x) = L$$

- The basic idea behind the notion of limit is this: We want to understand the behavior of a mathematical expression $f(x)$ *near* but not *at* the point $x = c$.
- The most practical use of this idea for this course is that limits can (sometimes) allow us to give meaning to mathematical expressions that evaluate to the meaningless form $0/0$.
- We will take three main approaches to understanding limits: (a) numerical, (b) graphical, and (c) algebraic. We will use the third approach most but the others should be understood.

Example: Numerical Approach.

Behavior of $f(x)$ for x near c

Consider the behavior of $f(x) = \dfrac{x^2 - 3x + 2}{x - 1}$ as x approaches 1.

x	0.8	0.9	0.99	1	1.01	1.1	1.2
$f(x)$	-1.2	-1.1	-1.01	*undefined*	-0.99	-0.9	-0.8

As x approaches 1, $f(x)$ approaches -1.

Definition

If $f(x)$ gets closer and closer to a number L as x gets closer and closer to c from both sides, then L is the *limit* of $f(x)$ as x approaches c. The behavior is expressed by

$$\lim_{x \to c} f(x) = L$$

Example: Graphical Approach.

It is important to remember that limits describes the behavior of a function *near* a particular point, not necessarily *at* the point itself. Note that the limit in each case is the same and is independent of the value of the function at $x = c$ or even if the function is defined at $x = c$.

Three functions for which $\lim_{x \to c} f(x) = L$

Example: Algebraic Approach.

We are looking at $\lim\limits_{x \to 1} \dfrac{x^2 - 3x + 2}{x - 1}$

- Note that as long as $x \neq 1$,

$$\frac{x^2 - 3x + 2}{x - 1} = \frac{(x-1)(x-2)}{x-1} = (x-2)$$

. Of course, if $x = 1$ then $\dfrac{x^2 - 3x + 2}{x - 1}$ is undefined.

- Therefore, $\lim\limits_{x \to 1} \dfrac{x^2 - 3x + 2}{x - 1} = \lim\limits_{x \to 1} (x - 2) = 1 - 2 = -1$ as we shall see.

Functions for which the limit does not exist

It is possible that the limit $\lim_{x \to c} f(x)$ does not exist.

Properties of Limits

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, then

$$\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

$$\lim_{x \to c}[f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$$

$$\lim_{x \to c}[kf(x)] = k \lim_{x \to c} f(x)$$

$$\lim_{x \to c}[f(x)g(x)] = [\lim_{x \to c} f(x)][\lim_{x \to c} g(x)]$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \quad \text{if } \lim_{x \to c} g(x) \neq 0$$

$$\lim_{x \to c}[f(x)]^p = [\lim_{x \to c} f(x)]^p \quad \text{if } [\lim_{x \to c} f(x)]^p \text{ exists}$$

Computation of Limits

Limits of Polynomials and Rational functions

If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \to c} p(x) = p(c) \quad \text{and} \quad \lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \quad \text{if } q(c) \neq 0$$

Example

Find $\lim_{x \to 2} (x^2 - 4x + 7)$.

Example

Find $\lim_{x \to 1} \frac{x+3}{2x+1}$.

Computation of Limits

Example

Find $\lim_{x \to 2} \dfrac{2x+3}{x-2}$.

Example

Find $\lim_{x \to 2} \dfrac{x^2+x-6}{x-2}$.

Limits involving Infinity

Limits at infinity

If the values of $f(x)$ approach the number L as x gets larger and larger,

$$\lim_{x \to +\infty} f(x) = L$$

If the values of $f(x)$ approach the number L as x gets larger and larger negatively,

$$\lim_{x \to -\infty} f(x) = M.$$

Graphically, $\lim_{x \to \pm\infty} f(x) = L$ means that $f(x)$ has a *horizontal asymptote* at the line $y = L$.

Limits involving Infinity

Example

If $k > 0$ and x^k is defined for all x, then for any constant A,

$$\lim_{x \to \pm\infty} \frac{A}{x^k} = 0.$$

Example

Find $\displaystyle\lim_{x \to +\infty} \frac{1 - 2x^3}{2x^3 - 5x + 4}$.

Example

Find $\displaystyle\lim_{x \to -\infty} \frac{x^2 + 2x - 3}{1 - 3x - x^3}$.

Limits involving Infinity

Infinite Limits

If $f(x)$ increases without bound as $x \to c$, we write

$$\lim_{x \to c} f(x) = +\infty.$$

If $f(x)$ decreases without bound as $x \to c$, then

$$\lim_{x \to c} f(x) = -\infty.$$

Graphically, $\lim_{x \to c} f(x) = \pm\infty$ means that $f(x)$ has a *vertical asymptote* at the line $x = c$.

Example

Find $\displaystyle\lim_{x \to -1/2} \frac{1 - 3x^3}{2x + 1}$.