Curve Sketching with the Second Derivative

Example

Determine where the function

$$f(x) = \frac{x^2}{x^2 + 3}$$

is increasing and decreasing, and where its graph is concave up and concave down. Find all relative extrema and points of inflection, and sketch the graph.

Quiz 7 : Sections 3.1, 3.2

$$f(x) = \frac{x^2}{x^2+3}$$

Intervals of increase / decrease

$$f'(x) = \frac{(x^2+3)(2x) - (x^2)(2x)}{(x^2+3)^2}$$

$$= \frac{\cancel{2x^3} + 6x - \cancel{2x^3}}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$$

Set $f'(x) = 0$. Solve. $6x = 0$

$x = 0 \leftarrow$ crit #

Critical point: $(0, f(0)) = (0, 0)$ //

decreasing — increasing

f': $- \; - \; - \; - \; - \; - \; 0 \; + \; + \; + \; + \; + \; +$

$-2 \quad -1 \quad 0 \quad 1 \quad 2$

$$f'(-1) = \frac{6(-1)}{((-1)^2+3)^2} < 0 \qquad f'(1) = \frac{6(1)}{(1^2+3)^2} > 0$$

$(0,0) \leftarrow$ local minimum

Concavity

$$f'(x) = \frac{6x}{(x^2+3)^2}$$

$$f''(x) = \frac{(x^2+3)^2(6) - (6x)(2(x^2+3)(2x))}{(x^2+3)^4}$$

$$= \frac{\cancel{(x^2+3)}\overbrace{[6(x^2+3) - 24x^2]}^{6x^2+18 \;-\; 24x^2}}{(x^2+3)^{\cancel{4}\,3}}$$

$$= \frac{-18x^2+18}{(x^2+3)^3} = \frac{-18(x^2-1)}{(x^2+3)^3}$$

Possible inflection points:

Set $f''=0$. Solve. $-18(x^2-1)=0$

$$x^2-1=0$$

$$(x+1)(x-1)=0$$

$$\underline{x=-1} \quad \underline{x=1}$$

f'': $- - - -$ 0 $+ + + + + +$ 0 $- - - - -$

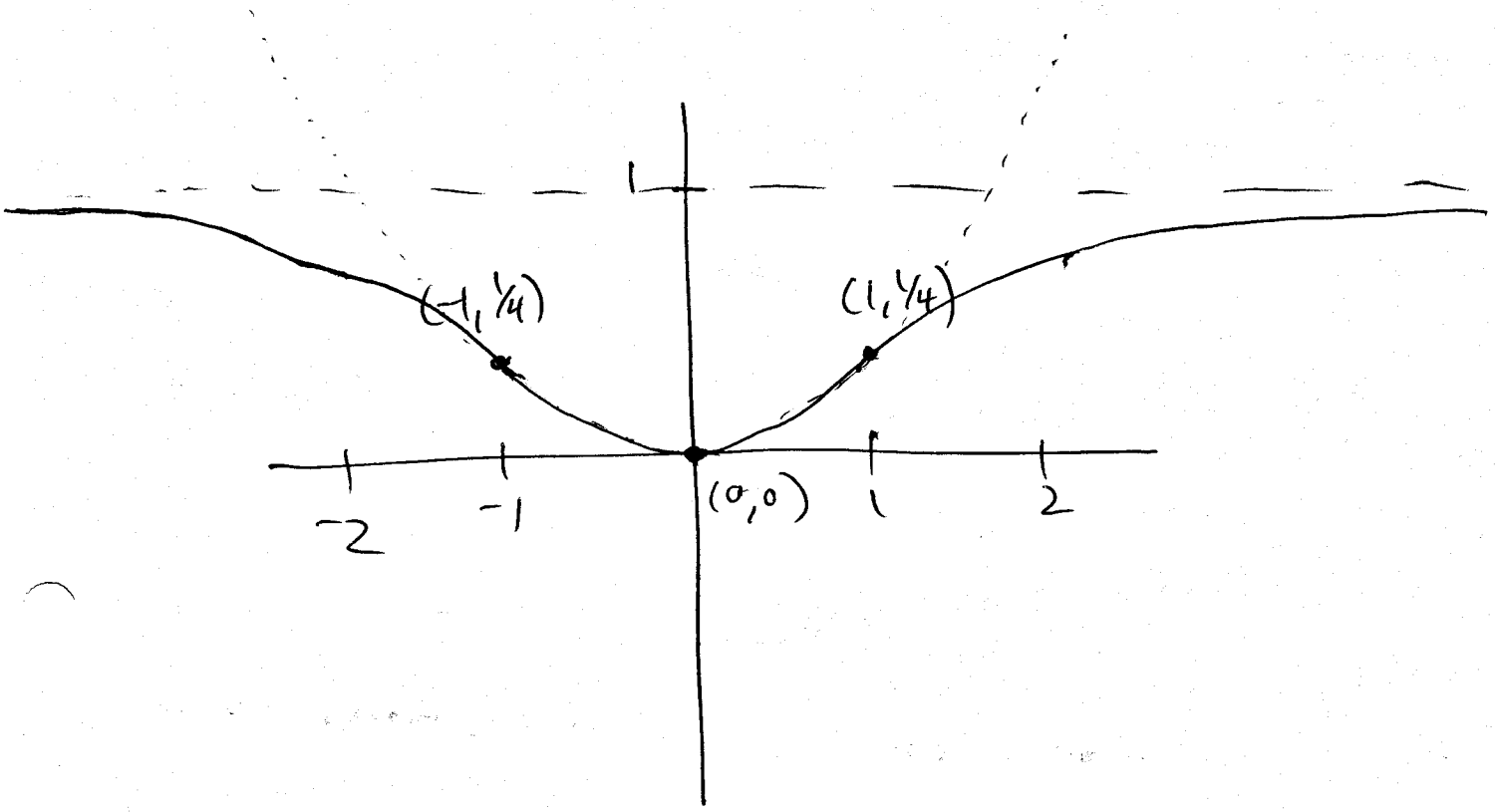
conc. down | conc. up | conc. down

-2 $\quad -1 \quad 0 \quad 1 \quad 2$

$$f''(-2) = \frac{(-18)(3)}{(4+3)^3} < 0 \qquad f''(0) = \frac{(-18)(-1)}{(3^3)} > 0 \qquad f''(2) = \frac{(-18)(3)}{(4+3)^3} < 0$$

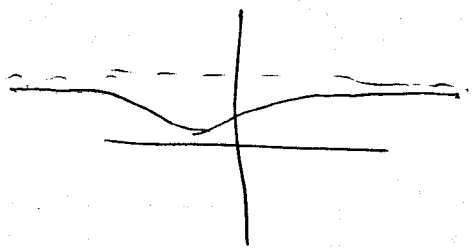
Inflection points: $(-1, f(-1))$, $(1, f(1))$

$(-1, \frac{1}{4})$ $(1, \frac{1}{4})$



Horizontal asymptotes:

$$\lim_{x \to \infty} \frac{x^2}{x^2+3} = 1$$



$$\lim_{x \to -\infty} \frac{x^2}{x^2+3} = 1$$

Concavity and Inflection Points

Example

The first derivative of a certain function $f(x)$ is

$$f'(x) = x^2 - 2x - 8.$$

(a) Find intervals on which f is increasing and decreasing.

(b) Find intervals on which the graph of f is concave up and concave down.

(c) Find the ~~y~~ x coordinate of the relative extrema and inflection points of f.

(a) Set $f' = 0$. Solve. $x^2 - 2x - 8 = 0$

$(x+2)(x-4) = 0$

$x = -2 \quad x = 4 \leftarrow$ crit #'s

$f'(-3) = (-1)(-7) > 0$

$f'(0) = (2)(-4) < 0$

f': + + + 0 − − − − − − − − − 0 + + +

INCR | DECR | INCR

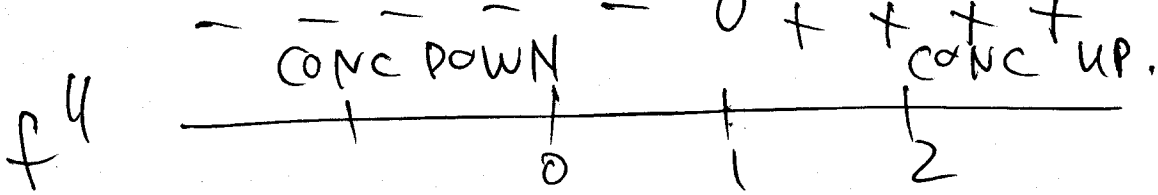
$-3 \quad -2 \quad 0 \quad 4 \quad 5$

$f'(5) = (7)(1) > 0$

(b) $f''(x) = 2x - 2$

Set $f''=0$. Solve. $2x-2=0$

$f''(0) = -2 < 0$ $f''(2) = 2 > 0$ $x = 1 \leftarrow$ poss. infl. pt.



(c) Local max at $(-2, ?)$

Local min at $(4, ?)$

Inflection point at $(1, ?)$

We can only get x-coordinates.

The Second Derivative Test

Suppose $f''(x)$ exists on an open interval containing $x = c$ and that $f'(c) = 0$.

- If $f''(c) > 0$, then f has a relative minimum at $x = c$.
- If $f''(c) < 0$, then f has a relative maximum at $x = c$.

However, if $f''(c) = 0$ or if $f''(c)$ does not exist, the test is inconclusive and f may have a relative maximum, a relative minimum, or no relative extremum at all at $x = c$.

The Second Derivative Test

Example

Find the critical points of

$$f(x) = x^3 + 3x^2 + 1$$

and use the second derivative test to classify each critical point as a relative maximum or minimum.

e.g. $f(x) = x^4$

Concavity: $f'(x) = 4x^3$ $\quad f''(x) = 12x^2$

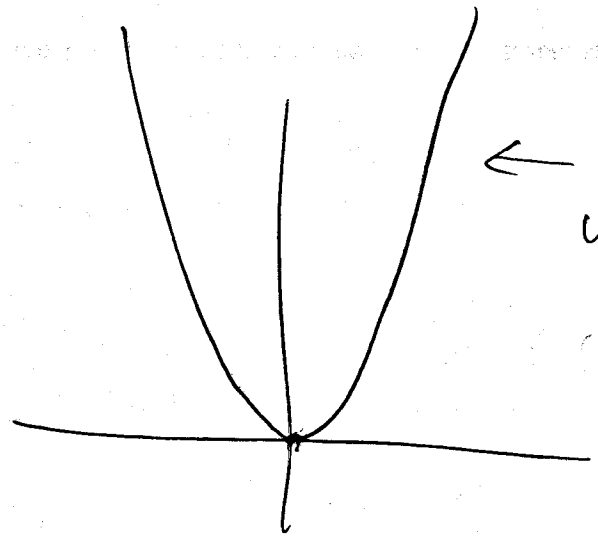
Set $f''=0$. Solve: $12x^2 = 0$

$x = 0 \leftarrow$

f'': $+ \quad + \quad + \quad + \quad 0 \quad + \quad + \quad + \quad +$

$-2 \quad -1 \quad 0 \quad 1 \quad 2$

$f''(-1) = 12 > 0 \qquad f''(1) = 12 > 0$



← concave up everywhere.

3.3. Curve Sketching

Vertical Asymptotes

The vertical line $x = c$ is a vertical asymptote of the graph of $f(x)$ if either

$$\lim_{x \to c^-} f(x) = +\infty \quad (\text{or } -\infty)$$

or

$$\lim_{x \to c^+} f(x) = +\infty \quad (\text{or } -\infty)$$

Check ~~at~~ X where denominator $= 0$.

Vertical Asymptotes

Example

Determine all vertical asymptotes of the graph of

$$g(x) = \frac{2x^2 + 2x}{x^2 - 3x - 4}$$

Vertical asymptotes:

$x^2 - 3x - 4 = 0$

$(x+1)(x-4) = 0$

$\underline{x = -1}$ $\underline{x = 4}$

vert. asymp.

Move work.

NOT A VERT. ASYMP. (HOLE IN GRAPH)

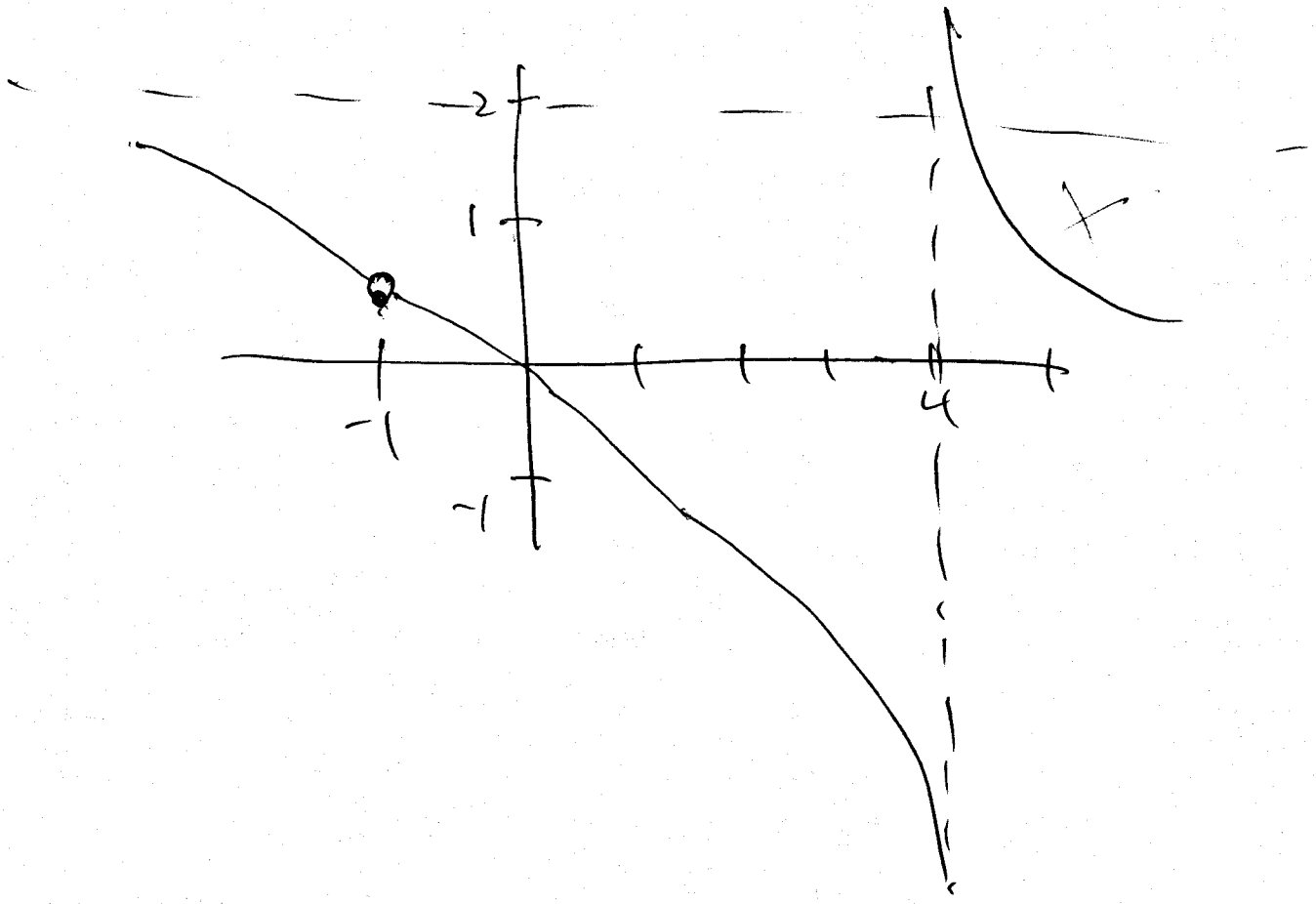
Check numerator:

$2(4)^2 + 2(4) = 40 \neq 0.$

$2(-1)^2 + 2(-1) = 2 - 2 = 0$

$$\lim_{x \to -1} \frac{2x^2 + 2x}{x^2 - 3x - 4} = \lim_{x \to -1} \frac{2x\cancel{(x+1)}}{\cancel{(x+1)}(x-4)}$$

$$= \frac{2}{5}$$


2
1
-1
-1
4

Horizontal Asymptotes

The horizontal line $y = b$ is a horizontal asymptote of the graph of $f(x)$ if

$$\lim_{x \to -\infty} f(x) = b$$

or

$$\lim_{x \to +\infty} f(x) = b$$

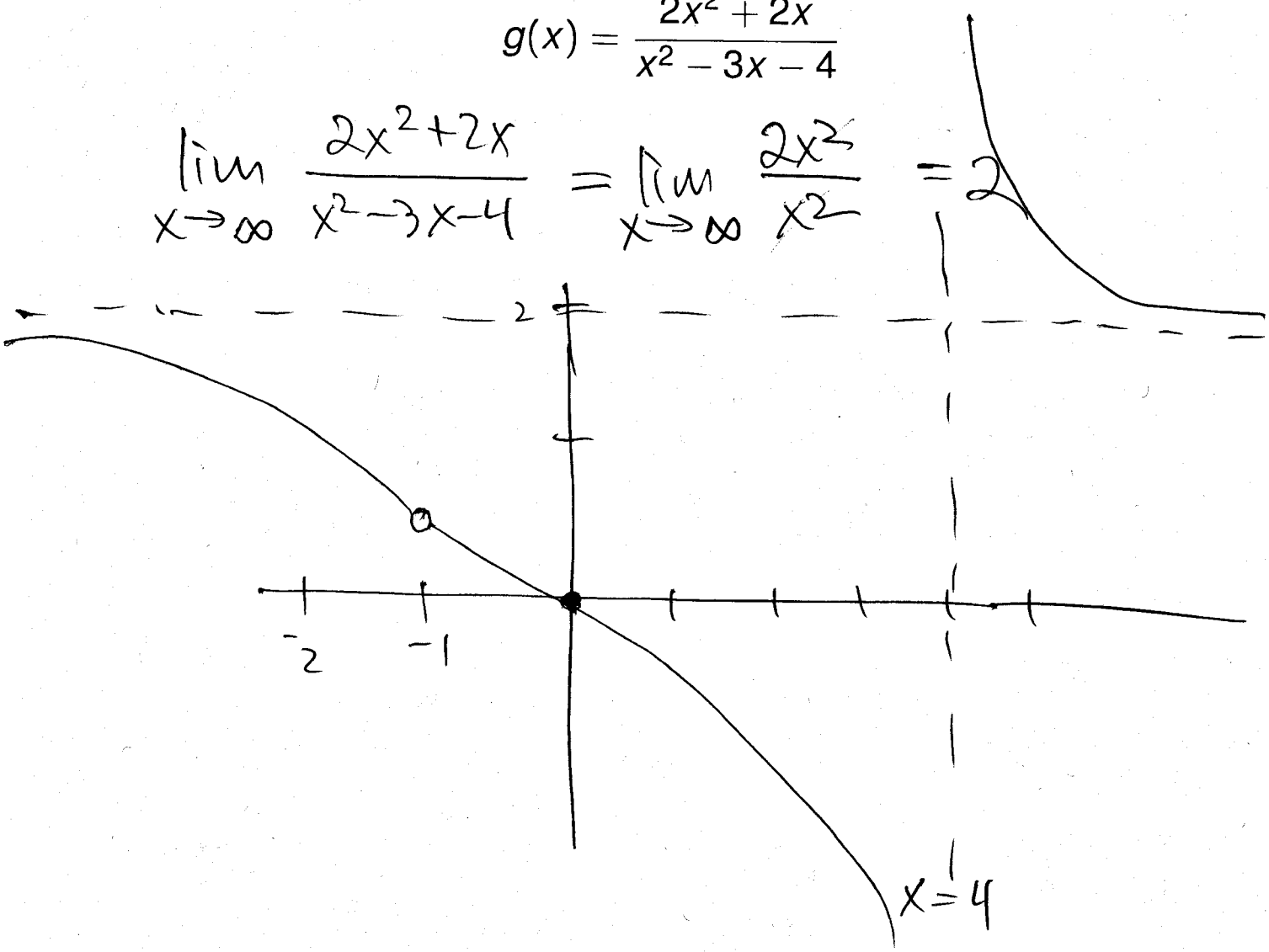
Horizontal Asymptotes

Example

Determine all horizontal asymptotes of the graph of

$$g(x) = \frac{2x^2 + 2x}{x^2 - 3x - 4}$$

$$\lim_{x \to \infty} \frac{2x^2+2x}{x^2-3x-4} = \lim_{x \to \infty} \frac{2x^2}{x^2} = 2$$



General Procedure for Sketching the Graph

Step 1. Find the domain of $f(x)$.

Step 2. Find and plot all intercepts.

Step 3. Determine all vertical and horizontal asymptotes and draw them.

Step 4. Find $f'(x)$ and determine the critical numbers and intervals of increase and decrease.

Step 5. Determine all relative extrema. Plot each relative maximum with a "cap" and each relative minimum with a "cup".

Step 6. Find $f''(x)$ and determine intervals of concavity and points of inflection. Plot inflection points with a "twist"

Step 7. Complete the sketch by joining the plotted points.

Curve Sketching

Example

Sketch the graph of $f(x) = \dfrac{4x}{(x+1)^2}$.

Vertical asymp at $x=-1$

Horizontal asymp at $y=0$

$$\boxed{\lim_{x\to\infty} \frac{4x}{(x+1)^2} = \lim_{x\to\infty} \frac{4x}{x^2} = \lim_{x\to\infty} \frac{4}{x} = 0}$$

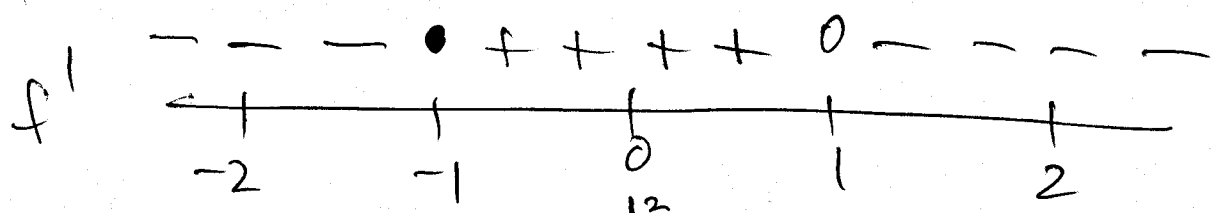
Increase / Decrease.

$$f'(x) = \frac{(x+1)^2(4) - (4x)(2(x+1))}{(x+1)^4}$$

$$= \frac{\cancel{(x+1)}[4(x+1)-8x]}{(x+1)^{\cancel{4}3}} = \frac{-4x+4}{(x+1)^3}$$

$$= \frac{-4(x-1)}{(x+1)^3}$$

Crit #s: $x = 1$, $x = -1$



$$f'(-2) = \frac{(-4)(-3)}{(-1)^3} < 0 \qquad f'(0) = \frac{(-4)(-1)}{(1)^3} > 0$$

$$f'(2) = \frac{(-4)(1)}{(3)^3} < 0$$

$\boxed{\text{Local max at } (1,1)}$

Concavity:

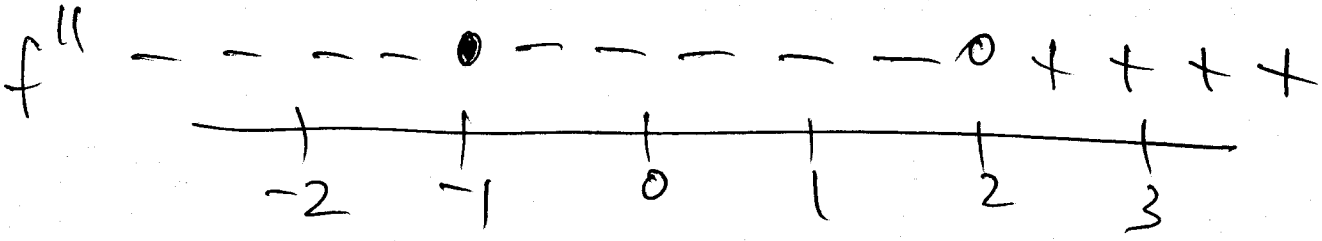
$$f''(x) = \frac{(x+1)^3(-4) - (-4(x-1))(3(x+1)^2)}{(x+1)^6}$$

$$= \frac{\cancel{(x+1)^2}[-4(x+1) - (-12(x-1))]}{(x+1)^{\cancel{6}4}}$$

$$= \frac{-4x - 4 + 12x - 12}{(x+1)^4} = \frac{8x - 16}{(x+1)^4}$$

$$= \frac{8(x-2)}{(x+1)^4} \;//$$

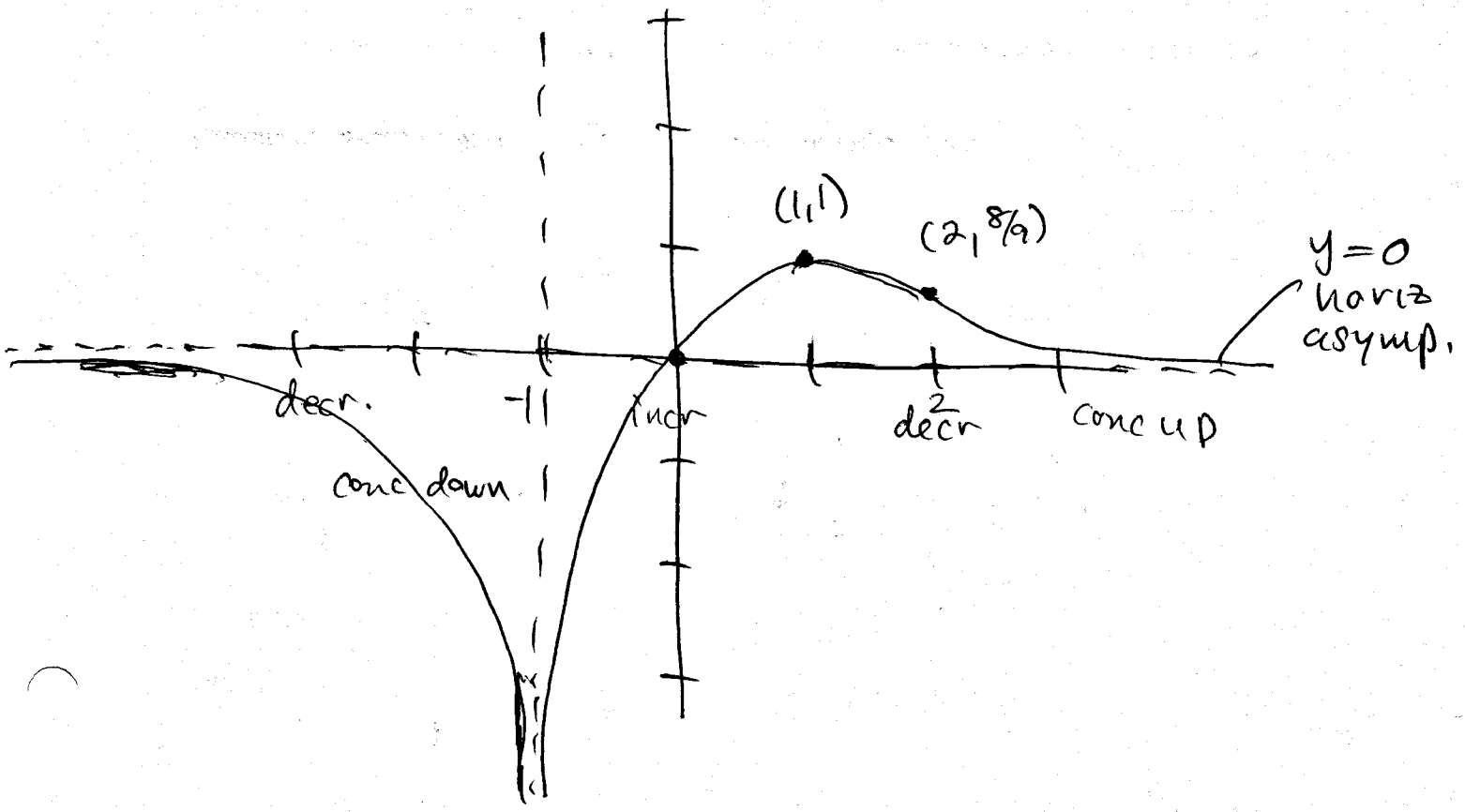
Possible inflection: $x=2$ $\quad x=-1$



$$f''(-2) = \frac{(8)(-4)}{(-1)^4} < 0 \qquad f''(0) = \frac{(8)(-2)}{(1)^4} < 0$$

$$f''(3) = \frac{(8)(1)}{(4)^4} > 0$$

$$\boxed{\text{Inflection point at: } \left(2, \frac{4(2)}{(2+1)^2}\right) = \left(2, \frac{8}{9}\right)}$$



Curve Sketching

Example

Sketch the graph of $f(x) = \dfrac{x+3}{x-5}$.

Curve Sketching

Example

Sketch the graph of $f(x) = \dfrac{x+1}{x^2+x+1}$.

3.4. Optimization

Absolute Maxima and Minima of a function

Let f be a function defined on an interval I containing the number c. Then

- $f(c)$ is the absolute maximum of f on I if $f(c) \geq f(x)$ for all x in I.
- $f(c)$ is the absolute minimum of f on I if $f(c) \leq f(x)$ for all x in I.

Collectively, absolute maxima and minima are called absolute extrema.

Absolute Extrema on a Closed interval

How to Find the Absolute Extrema of a Continuous Function f on $a \leq x \leq b$

Step 1. Find all critical numbers of f in $a < x < b$.

Step 2. Compute $f(x)$ at the critical numbers found in step 1 and at the endpoints $x = a$ and $x = b$.

Step 3. The largest and smallest values found in step 2 are, respectively, the absolute maximum and absolute minimum values of $f(x)$ on $a \leq x \leq b$.

Absolute Extrema on a Closed interval

Example

21 1

Find the absolute maximum and absolute minimum (if any) of

$$f(x) = x^3 + 3x^2 + 1; \quad \boxed{-3 \le x \le 2.}$$

$f'(x) = 3x^2 + 6x$

$3x^2 + 6x = 0$

$3x(x+2) = 0$

$x = 0 \quad x = -2$

$-8 + 12 + 1 = 5$

$-27 + 27 + 1$

$8 + 12 + 1 = 21$

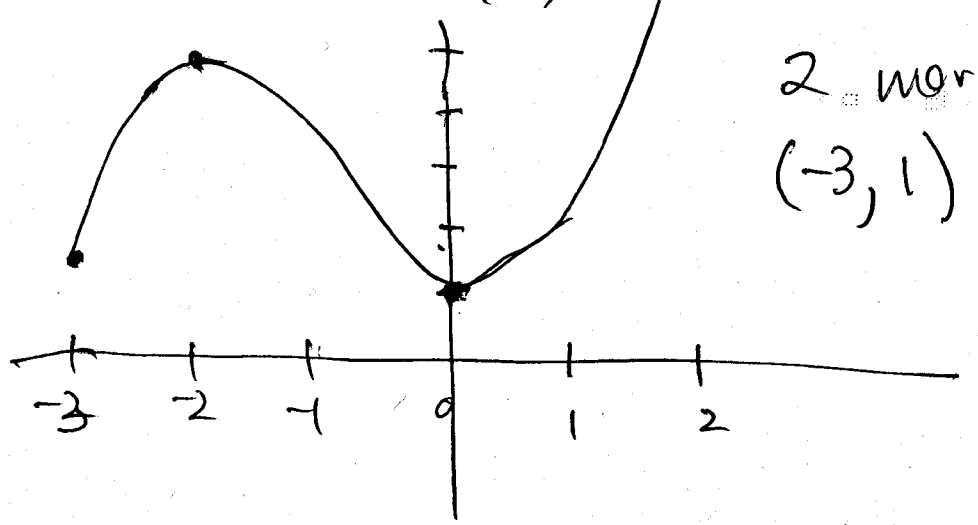
f': $+ + + \; 0 \; - - - - \; 0 \; + + + + +$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2$

Crit. points: $(-2, 5)$ $(0, 1)$

2 more points:

$(-3, 1)$ $(2, 21)$



Absolute Extrema on a Closed interval

Example

Find the absolute maximum and absolute minimum (if any) of

$$f(t) = \frac{t^2}{t-1}; \quad -2 \leq t \leq \frac{1}{2}.$$

Absolute Extrema on a general interval

Example

Find the absolute maximum and absolute minimum (if any) of

$$f(u) = u + \frac{16}{u}; \quad u > 0.$$