Exponentials + Logarithms

$f(x) = b^x \quad b > 0,\ b \neq 1$

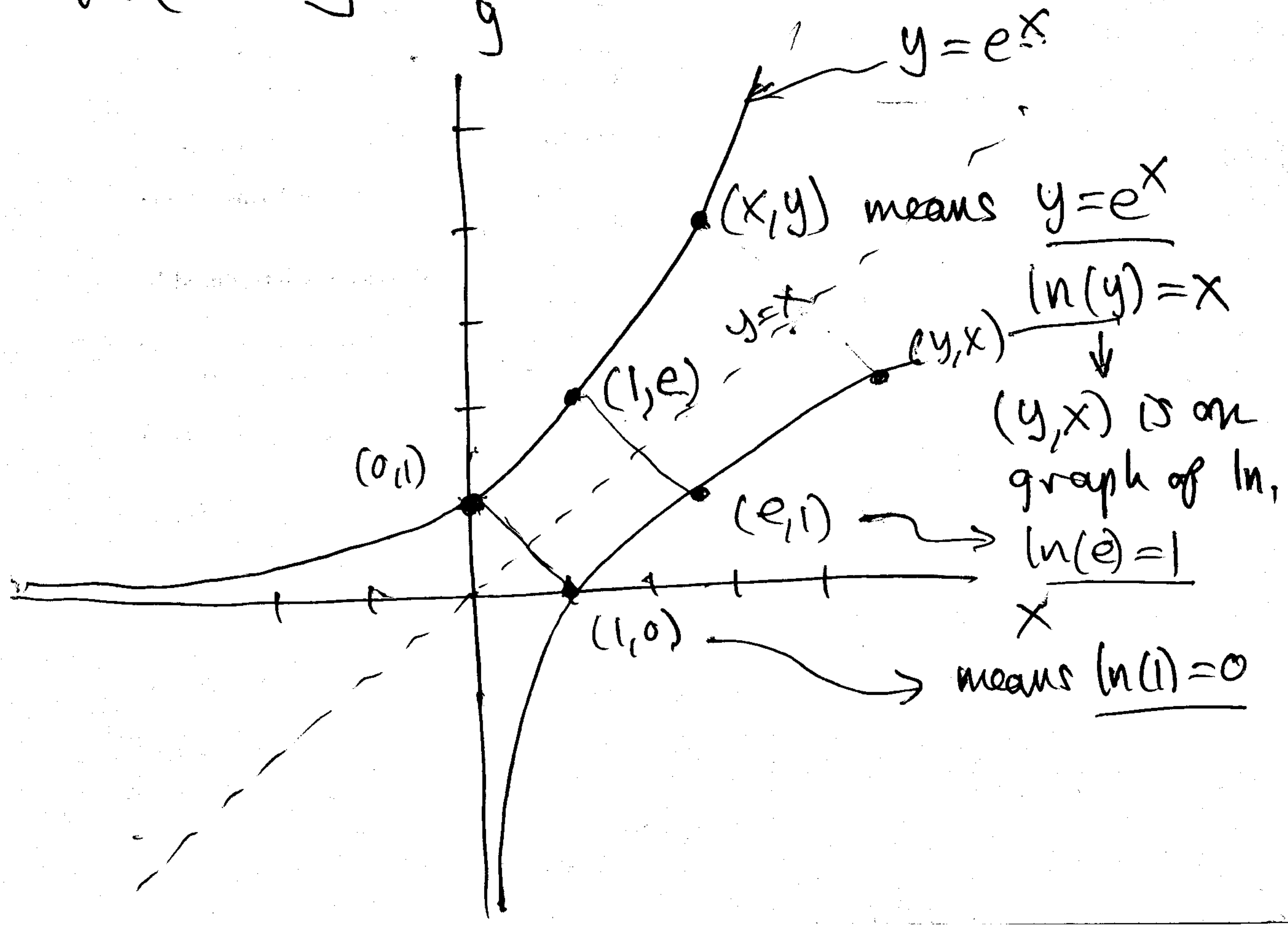
$f(x) = \log_b(x)$

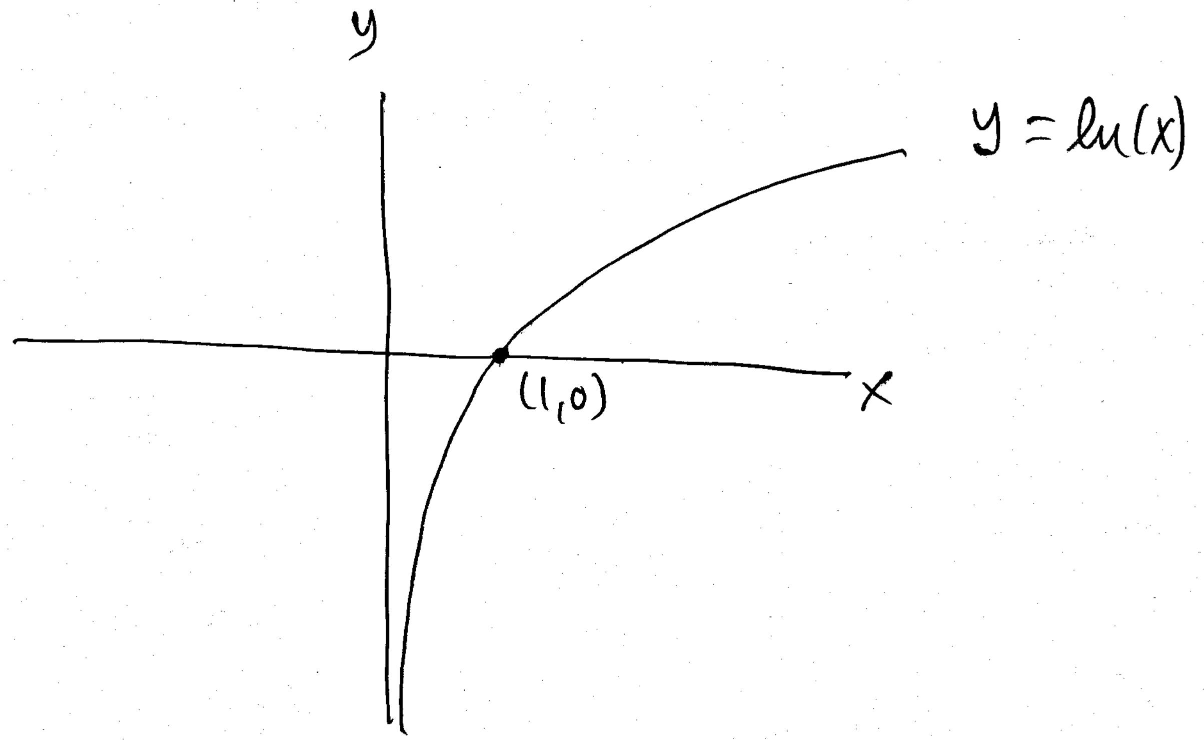
$\log_b(x) = y \iff b^y = x$

1. Graph of $\log_b(x)$.

Will look at $\ln(x) = \log_e(x)$

$\ln(x) = y \iff e^y = x$





2. Conversion Formula.

Calculator $\boxed{\text{LN}}$ $\cancel{\boxed{\text{LOG}}}$

$\uparrow$ $\log_e$ $\uparrow$ $\log_{10}$

How would you find $\log_3(8)$?

$\log_3(8) = y$

$3^y = 8$

$\ln(3^y) = \ln(8)$

$y \ln(3) = \ln(8)$

$\rightarrow y = \dfrac{\ln(8)}{\ln(3)} = \log_3(8)$

Conversion Formula for Logarithms

If a and b are positive numbers with $b \neq 1$, then

$$\log_b a = \frac{\ln a}{\ln b}$$

Example

Find $\log_5 3$.

$$\log_5(3) = \frac{\ln(3)}{\ln(5)} \approx .6826$$

$$\log_5(3) \cong .6826$$

$$5^{.6826} \cong 3$$

Think about $\frac{d}{dx}(b^x)$ $b>0, b\neq 1$

$$\frac{d}{dx}(b^x) = \lim_{h\to 0} \frac{b^{x+h}-b^x}{h} = \lim_{h\to 0} \frac{b^x b^h - b^x}{h}$$

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

$f(x) = b^x$

$$= \lim_{h\to 0} \frac{b^x(b^h-1)}{h} = b^x \left(\lim_{h\to 0} \frac{b^h-1}{h}\right)$$

some (magic) number

Q: Is there a b so that $\lim_{h\to 0} \frac{b^h-1}{h} = 1$?

A: if $\frac{b^h-1}{h} = 1 \rightarrow b^h - 1 = h \rightarrow b^h = 1+h$

$\longrightarrow b = (1+h)^{1/h}$ The true solution is: $\lim_{h\to 0}(1+h)^{1/h} = e$

4.3 Derivatives of b^x or $\log_b(x)$,

Differentiation of Exponential Functions

The Derivative of the Exponential Function

$$\boxed{\frac{d}{dx}(e^x) = e^x \quad \text{for every real number} \quad x}$$

Example

Differentiate the function $f(x) = \dfrac{e^x}{x}$.

$$f'(x) = \frac{x \cdot e^x - e^x \cdot 1}{x^2} = \frac{e^x(x-1)}{x^2}$$

OR $\quad f(x) = e^x(x^{-1})$

$$f'(x) = e^x(-x^{-2}) + (x^{-1})(e^x)$$

$$= e^x(x^{-1} - x^{-2})$$

$$= e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) = e^x\left(\frac{x-1}{x^2}\right)$$

Differentiation of Exponential Functions

The Chain Rule for Exponential Functions
If $u(x)$ is a differentiable function of x, then

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx} e^{u(x)} = \underbrace{e^{u(x)}}_{f'(g(x))} \underbrace{u'(x)}_{g'(x)}$$

$$\frac{d}{dx}\left(e^{g(x)}\right)$$

Example
Differentiate the function $f(x) = xe^{2x}$.

$$\begin{aligned} f'(x) &= x \frac{d}{dx}\left(e^{2x}\right) + e^{2x}(1) \\ &= x\left(e^{2x} \cdot \frac{d}{dx}(2x)\right) + e^{2x} \\ &= x\left(e^{2x} \cdot 2\right) + e^{2x} \\ &= e^{2x}(2x+1) \end{aligned}$$

Differentiation of Exponential Functions

Example

Find the largest and smallest values of the function $F(x) = e^{x^2-2x}$ over the closed interval $0 \leq x \leq 2$.

Find critical numbers.

$$F'(x) = \left(e^{x^2-2x}\right)\frac{d}{dx}(x^2-2x)$$

$$= e^{x^2-2x}(2x-2)$$

$$e^{x^2-2x}(2x-2) = 0$$

$$2x-2 = 0$$

$$x = 1$$

$$F(0) = e^{(0)^2-2(0)} = e^0 = 1$$

$$F(1) = e^{(1)^2-2(1)} = e^{-1}$$

$$F(2) = e^{(2)^2-2(2)} = e^0 = 1$$

Smallest value: $e^{-1} \approx .37$

Largest value: 1.

4.3. Differentiation of Logarithmic and Exponential Functions

Derivative of $\ln x$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x} \quad \text{for} \quad x > 0}$$

Example

Differentiate the function $f(x) = x \ln \sqrt{x}$.

$$f(x) = x \ln(\sqrt{x}) = x \ln(x^{1/2})$$

$$= \frac{1}{2}x \ln(x)$$

$$f'(x) = \left(\frac{1}{2}x\right) \frac{d}{dx} \ln(x) + \ln(x) \left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\cancel{x} \cdot \frac{1}{\cancel{x}} + \frac{1}{2}\ln(x)$$

$$= \frac{1}{2} + \frac{1}{2} \ln(x).$$

Differentiation of Logarithmic Functions

The Chain Rule for Logarithmic Functions $\frac{d}{dx} f(g(x))$

If $u(x)$ is a differentiable function of x, then

$$\boxed{\frac{d}{dx}[\ln u(x)] = \frac{u'(x)}{u(x)}}$$

$$= f'(g(x)) \cdot g'(x)$$

$$f(g(x)) = \ln(g(x))$$

$$\frac{1}{u(x)} \cdot u'(x) = \frac{u'(x)}{u(x)},$$

$$f'(g(x)) \quad g'(x)$$

Example

Differentiate the function $f(x) = \ln(x^2 + 1)$.

$$f'(x) = \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1) = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$

Differentiation of Logarithmic Functions

Example

Differentiate the function $f(x) = \ln(x^3 - 5x + 4)$.

$$f'(x) = \frac{1}{x^3-5x+4} \frac{d}{dx}(x^3-5x+4)$$

$$= \frac{3x^2-5}{x^3-5x+4}$$

Differentiation of Logarithmic Functions

Example

Find an equation for the tangent line to $y = x + \ln x$ at the point where $x = e$.

Slope: $y = x + \ln(x)$

$$\frac{dy}{dx} = 1 + \frac{1}{x}$$

$$\left.\frac{dy}{dx}\right|_{x=e} = 1 + \frac{1}{e} \approx 1.37$$

Point: $(e, e + \ln(e)) = (e, e+1)$

$$y - (e+1) = \left(1 + \frac{1}{e}\right)(x - e)$$

$$y = \left(1+\frac{1}{e}\right)x - \underbrace{e\left(1+\frac{1}{e}\right)}_{(e+1)} + e + 1$$

$$= \left(1+\frac{1}{e}\right)x$$

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Logarithmic Differentiation

Differentiating a function that involves products, quotients, or powers can often be simplified by first taking the logarithm of the function.

Step 1. Take logarithms of both sides of the expression for $f(x)$ and simplify the resulting equation.

Step 2. Use the chain rule to differentiate both sides.

Step 3. Multiply both sides with $f(x)$ to get $f'(x)$.

Logarithmic Differentiation

Example

Use logarithmic differentiation to find the derivative of

$f(x) = \sqrt[4]{\frac{2x+1}{1-3x}}$.

$$\ln(f(x)) = \ln\left(\left(\frac{2x+1}{1-3x}\right)^{1/4}\right)$$

$$= \frac{1}{4}\ln\left(\frac{2x+1}{1-3x}\right)$$

$$= \frac{1}{4}\left(\ln(2x+1) - \ln(1-3x)\right)$$

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{4}\left(\frac{2}{2x+1} - \frac{-3}{1-3x}\right)$$

$$f(x)\,\frac{f'(x)}{f(x)} = \frac{1}{4}\left(\frac{2}{2x+1} + \frac{3}{1-3x}\right)(f(x))$$

$$f'(x) = \frac{1}{4}\left(\frac{2}{2x+1} + \frac{3}{1-3x}\right)\left(\frac{2x+1}{1-3x}\right)^{1/4} \quad //$$

Logarithmic Differentiation

Example

Use logarithmic differentiation to find the derivative of
$f(x) = \dfrac{e^{3x}(x^2+5)}{(1-x)^5}$.