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Distributions and measures on the boundary of a tree

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Abstract

In this paper, we analyze the space $\mathcal{D}$ of distributions on the boundary Ω of a tree and its subspace $\mathcal{B}_0$, which was introduced in [Amer. J. Math. 124 (2002) 999–1043] in the homogeneous case for the purpose of studying the boundary behavior of polyharmonic functions. We show that if $\mu \in \mathcal{B}_0$, then μ is a measure which is absolutely continuous with respect to the natural probability measure λ on Ω, but on the other hand there are measures absolutely continuous with respect to λ which are not in $\mathcal{B}_0$. We then give the definition of an absolutely summable distribution and prove that a distribution can be extended to a complex measure on the Borel sets of Ω if and only if it is absolutely summable. This is also equivalent to the condition that the distribution have finite total variation. Finally, we show that for a distribution μ, Ω decomposes into two subspaces. On one of them, a union of intervals A_μ, μ restricted to any finite union of intervals extends to a complex measure and on A_μ we give a version of the Jordan, Hahn, and Lebesgue–Radon–Nikodym decomposition theorems. We also show that there is no interval in the complement of A_μ in which any type of decomposition theorem is possible. All the results in this article can be generalized to results on good (in particular, compact infinite) ultrametric spaces, for example, on the p-adic integers and the p-adic rationals.

Keywords: Distributions; Measures; Trees

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1. Introduction

In the classical sense, a distribution is an element of the dual of the space of C^∞-functions on a manifold, whereas a complex measure may be thought of as an element of the dual of the space of continuous functions (using the Riesz representation theorem). Thus, a measure is a distribution, but not vice versa. For example, the distribution $f \mapsto f'(x_0)$ (for a fixed x_0) cannot be extended to a measure.

Distributions and measures on the boundary play an important role in the harmonic analysis of trees. Although in this setting measures are the usual complex measures, distributions must be conceived differently, since the boundary of a tree is not a manifold; indeed, it is an arbitrary compact ultrametric space.

In this paper, we shall study some classes of distributions, relating them to one another and to the space of Borel measures. Among other results, we define a simple condition on distributions, which we call *absolute summability*, and show that the space of measures is exactly the space of the distributions that are absolutely summable. We use [2] as a general reference on trees.

A *tree* is a locally finite connected graph with no loops, which, as a set, we identify with the collection of its vertices. Two vertices v and w of a tree are called *neighbors* if there is an edge connecting them, in which case we use the notation $v \sim w$. A *path* is a finite or infinite sequence of vertices $[v_0, v_1, \ldots]$ such that $v_k \sim v_{k+1}$ and $v_{k-1} \neq v_{k+1}$ for all k. If u and v are any vertices, we denote by $[u, v]$ the unique path joining them.

Fixing a vertex $e \in T$ as a root of the tree, the *predecessor* u^- of a vertex u, with $u \neq e$, is the next to the last vertex of the path from e to u. An *ancestor* of u is any vertex in the path from e to u^-. A vertex u is a *descendant* of v if v is an ancestor of u. By convention, we set $e^- = e$. We call *children* of a vertex v the vertices u such that $u^- = v$. A vertex is said to be *terminal* if it has only one neighbor.

A tree T may be endowed with a metric d as follows. If u, v are vertices, $d(u, v)$ is the number of edges in the unique path from u to v. Given a root e, the *length* of a vertex v is defined as $|v| = d(e, v)$. For $v \in T$ and an integer $n > |v|$, let $D_n(v)$ be the set of descendants of v of length n.

A *nearest-neighbor transition probability* on the vertices of a tree T is a function on $T \times T$ such that $p(v, u) > 0$, if v and u are neighbors, $p(v, u) = 0$, if v and u are not neighbors, and $\sum_{u \sim v} p(v, u) = 1$ for each vertex v.

By a *homogeneous tree* of degree $q + 1$ (with $q \geqslant 2$) we mean a tree all of whose vertices have exactly $q + 1$ neighbors.

A transition probability p on a tree is said to be *isotropic* if for any pair of vertices u and v, $p(v, u)$ depends only on v and, in particular, is equal to the reciprocal of the number of neighbors of v when $u \sim v$.

The *boundary* Ω of T is the union of the set of terminal vertices and the set of equivalence classes of infinite paths under the relation $\simeq$ defined by the shift, $[v_0, v_1, \ldots] \simeq [v_1, v_2, \ldots]$. For any vertex u, we denote by $[u, \omega)$ the (unique) path starting at u in the class ω; then Ω can be identified with the set of paths starting at u. Furthermore, Ω is a compact space under the topology generated by the sets

$$I_v = \{\omega \in \Omega\colon\ v \in [e, \omega)\},$$

which we call *intervals*. Clearly, $\Omega = I_e$. For $v \in T$, $n \in \mathbb{N}$, with $n \leqslant |v|$, define v_n to be the vertex of length n in the path $[e, v]$. Similarly, for a class ω the path $[e, \omega)$ will be denoted by $[\omega_0, \omega_1, \omega_2, \ldots]$.

If u is a vertex and $\omega = [u = \omega_0, \omega_1, \ldots]$ and $\omega' = [u = \omega'_0, \omega'_1, \ldots]$ are distinct boundary points, define $\omega \wedge \omega' = \omega_k$, where k is the largest integer such that $\omega_k = \omega'_k$.

We shall assume that Ω is infinite, because there is no difference between a measure and a distribution on a finite set.

Observation 1.1. *The complement of a finite union of intervals is again a finite union of intervals. To see this observe that any finite union of intervals can be written as $\bigcup_{k=1}^{m} I_{w_k}$ where the vertices w_k have fixed length N for some $N \in \mathbb{N}$. Letting $W = \{w_1, \ldots, w_m\}$, we see that the complement of this set is $\bigcup_{|v|=N,\, v \notin W} I_v$, a finite union.*

Define $C^\infty(\Omega)$ to be the algebra generated by the characteristic functions of the intervals. So $C^\infty(\Omega)$ is the set of finite linear combinations of the functions $\xi_v : \Omega \to \mathbb{C}$ defined by

$$\xi_v(\omega) = \begin{cases} 1 & \text{if } \omega_{|v|} = v, \\ 0 & \text{otherwise,} \end{cases}$$

where $[e, \omega) = [\omega_0, \omega_1, \ldots]$.

A *distribution* is an element of the dual $\mathcal{D}$ of the space $C^\infty(\Omega)$. Equivalently, a distribution is a finitely additive complex-valued set function defined on finite unions of the sets I_v. A complex measure is a countably additive complex-valued set function defined on the σ-algebra generated by the sets I_v (in particular, it is finite-valued). Thus a complex measure restricts to a distribution, but not every distribution extends to a complex measure, as seen in the following example. For completeness we also point out that a signed measure is a countably additive set function with values either in $\mathbb{R} \cup \{\infty\}$ or in $\mathbb{R} \cup \{-\infty\}$. A positive measure is a signed measure attaining its values in $[0, \infty]$.

Example 1.1. Let T be any tree rooted at e containing an infinite path $\{e = w_0, w_1, w_2, \ldots\}$ such that each w_n has at least three neighbors. For all $n \geqslant 1$, let v_n be a neighbor of w_{n-1} other than w_{n-2} and w_n, and pick a path $[v_n = v_{n,n}, v_{n,n+1}, \ldots]$ either infinite or ending at a terminal vertex, with $|v_{n,i}| = i$. Define a distribution μ on Ω by setting $\mu(\Omega) = 0$, $\mu(I_{v_{n,i}}) = 1$, $\mu(I_{w_n}) = -n$, $\mu(I_u) = 0$ if $u \neq w_n, v_{n,i}$, for all n and i. By construction, the intervals I_{v_n} are pairwise disjoint and μ is finitely additive on all intervals. Thus $\mu \in \mathcal{D}$, but cannot be extended to a complex measure since otherwise its extension would satisfy

$$\mu\Bigl(\coprod I_{v_n}\Bigr) = \sum_n \mu(I_{v_n}) = \infty,$$

where the symbol $\coprod$ stands for disjoint union. On the other hand, μ cannot be extended to a signed measure either since otherwise the class ω defined by the path $[w_0, w_1, w_2, \ldots]$ would satisfy $\mu(\{\omega\}) = \lim_{n\to\infty} \mu(I_{w_n}) = -\infty$. So μ would attain the values ∞ and $-\infty$.

As is customary, a function on a tree T will mean a function on its set of vertices. If p is a nearest-neighbor transition probability on T, the *Laplacian* of a function $f : T \to \mathbb{C}$ is defined as $\Delta f = \mu_1 f - f$, where

$$\mu_1 f(v) = \sum_{u \sim v} p(v, u) f(u).$$

A function f on T is said to be *harmonic* if its Laplacian is identically zero.

In this work, we shall assume that T is a tree (not necessarily homogeneous) rooted at e endowed with an isotropic transition probability. By [2], there is a one-to-one correspondence between the harmonic functions on T and the distributions on Ω.

Observe that if $[v_0, v_1, \ldots]$ is an infinite path starting at e, then

$$I_{v_j} - I_{v_{j+1}} = \coprod_{u \in C_j} I_u,$$

where C_j is the set consisting of the children of v_j unequal to v_{j+1}. Thus, for each $v \in T$ with $|v| = n$, we have

$$\Omega = \bigcup_{j=0}^{n-1} (I_{v_j} - I_{v_{j+1}}) \cup I_v, \tag{1}$$

a finite disjoint union of intervals.

Denote by λ the *Lebesgue measure* on Ω: if for any $v \in T$, q_v is the number of forward neighbors of v, then

$$\lambda(I_v) = \frac{1}{c_v}, \quad \text{where } c_v = \begin{cases} \prod_{j=0}^{|v|-1} q_{v_j} & \text{if } v \neq e, \\ 1 & \text{if } v = e. \end{cases}$$

The Lebesgue measure is the unique probability measure on Ω such that $\lambda(I_v)$ is divided evenly among the intervals defined by all the forward neighbors of v.

Definition 1.1. We say that a distribution μ on Ω is *absolutely summable* if for any countable collection $\{I_n\}$ of pairwise disjoint intervals, $\sum \mu(I_n)$ is absolutely convergent.

It is easy to see that a nonnegative distribution is absolutely summable: if $\{I_n\}$ is a finite collection of intervals, then its complement in Ω is a finite union of intervals J_m. Thus

$$\sum \mu(I_n) \leqslant \sum \mu(I_n) + \sum \mu(J_m) = \mu(\Omega) < \infty.$$

Thus if $\{I_n\}$ is an infinite collection of pairwise disjoint intervals, $\mu(\Omega)$ is an upper bound for $\sum \mu(I_n)$.

Let $\mathcal{M}$ be the space of complex (Borel) measures on Ω and let $\mathcal{M}_{\mathrm{AC}}$ be the subspace of measures which are absolutely continuous with respect to λ. Let $\mathcal{D}_{\mathrm{AS}}$ be the space of distributions which are absolutely summable, and let $\mathcal{D}_{\mathrm{FTV}}$ be the space of distributions with finite total variation (see Definition 3.3).

Definition 1.2. Let $\mathcal{B}_0$ be the space of distributions ν on Ω satisfying the condition

$$\sum_{v \neq e} \left| \nu(I_v) - \frac{1}{q_{v^-}} \nu(I_{v^-}) \right| < \infty.$$

Notice that $\lambda(I_v) = \lambda(I_{v^-})/q_{v^-}$, so that the above sum measures how far ν is from being a multiple of λ.

We summarize the results of this paper by

$$\mathcal{B}_0 \subsetneqq \mathcal{M}_{\mathrm{AC}} \subsetneqq \mathcal{M} = \mathcal{D}_{\mathrm{AS}} = \mathcal{D}_{\mathrm{FTV}} \subsetneqq \mathcal{D}. \tag{2}$$

The results that are most difficult to prove are the proper containment of $\mathcal{B}_0$ in $\mathcal{M}_{\mathrm{AC}}$ (see Theorem 2.1) and the equality $\mathcal{M} = \mathcal{D}_{\mathrm{AS}} = \mathcal{D}_{\mathrm{FTV}}$ (see Theorem 3.2).

All the results of this paper translate directly to good ultrametric spaces, for example, the p-adic integers or the p-adic rationals. All the details are shown in Section 5.

2. The space $\mathcal{B}_0$

In [5], we introduced the space $\mathcal{B}_\alpha$, for all $\alpha \geqslant 0$, for the purpose of studying the boundary behavior of polyharmonic functions on homogeneous trees of degree $q+1$. The space $\mathcal{B}_\alpha$ turns out to be precisely the Besov–Lipschitz space $B^\alpha_{1,1}$ defined in [4] which can be identified with the space of distributions ν such that

$$\sum_{v \neq e} q^{\alpha|v|} \left| \nu(I_v) - \frac{1}{q} \nu(I_{v^-}) \right| < \infty.$$

Proposition 2.1. *Let $\epsilon : T \to \mathbb{C}$ be such that*

$$\sum_{w^- = v} \epsilon(w) = 0 \quad \textit{for all } v \in T. \tag{3}$$

Define

$$\mu(I_v) = \sum_{j=0}^{|v|} \frac{\epsilon(v_j)}{\prod_{k=j}^{|v|-1} q_{v_k}}, \tag{4}$$

where we recall $[v_0, v_1, \ldots, v_{|v|}]$ is the path from e to v. Then $\mu \in \mathcal{D}$. Conversely, if $\mu \in \mathcal{D}$, then the function ϵ defined by

$$\epsilon(v) = \begin{cases} \mu(I_v) - \frac{1}{q_{v^-}} \mu(I_{v^-}) & \textit{if } v \neq e, \\ \mu(I_e) & \textit{if } v = e, \end{cases} \tag{5}$$

satisfies (3) *and* (4).

Proof. Assume first ϵ satisfies condition (3) and let μ be defined as in (4). To prove that $\mu \in \mathcal{D}$, it suffices to show that

$$\sum_{w^- = v} \mu(I_w) = \mu(I_v).$$

But for $w^- = v$,

$$\mu(I_w) = \frac{1}{q_v} \mu(I_v) + \epsilon(w).$$

Thus

$$\sum_{w^-=v} \mu(I_w) = \sum_{w^-=v} \frac{1}{q_v}\mu(I_v) + \sum_{w^-=v} \epsilon(w) = \mu(I_v).$$

For the converse, assume $\mu \in \mathcal{D}$ and ϵ satisfies (5). Then for $v \in T$, we have

$$\sum_{w^-=v}\left[\mu(I_w) - \frac{1}{q_v}\mu(I_v)\right] = \sum_{w^-=v} \mu(I_w) - \mu(I_v) = 0$$

and

$$\sum_{j=0}^{|v|} \frac{\epsilon(v_j)}{\prod_{k=j}^{|v|-1} q_{v_k}} = \sum_{j=1}^{|v|} \frac{\mu(I_{v_j}) - \mu(I_{v_{j-1}})/q_{v_{j-1}}}{\prod_{k=j}^{|v|-1} q_{v_k}} + \frac{\mu(I_e)}{\prod_{k=0}^{|v|-1} q_{v_k}} = \mu(I_v),$$

completing the proof. □

It now follows that $\mu \in \mathcal{B}_0$ if and only if the corresponding function ϵ on T satisfies the condition $\sum_{v\in T} |\epsilon(v)| < \infty$.

Theorem 2.1. (a) *If $\mu \in \mathcal{B}_0$, then μ can be extended to a complex measure on the Borel sets of Ω which is absolutely continuous with respect to λ.*

(b) *There exist measures which are absolutely continuous with respect to λ but are not in $\mathcal{B}_0$.*

Proof. Assume $\mu \in \mathcal{B}_0$. Let ϵ be the function corresponding to μ in Proposition 2.1. Observe that if $\omega \in \Omega$ and $[e, \omega) = [\omega_0, \omega_1, \omega_2, \ldots]$, we have

$$\epsilon(\omega_k) = \mu(I_{\omega_k}) - \frac{1}{q_{\omega_{k-1}}}\mu(I_{\omega_{k-1}}).$$

Thus

$$c_v\mu(I_v) = \sum_{n=1}^{|v|}\left[c_{v_n}\mu(I_{v_n}) - c_{v_{n-1}}\mu(I_{v_{n-1}})\right] + \mu(I_e),$$

whence

$$c_v\mu(I_v) - \mu(I_e) = \sum_{n=1}^{|v|} c_{v_n}\epsilon(v_n).$$

Consequently, $\lim_{n\to\infty} c_{\omega_n}\mu(I_{\omega_n})$ exists if and only if $\sum_{n=0}^{\infty} c_{\omega_n}\epsilon(\omega_n)$ exists. To show that this is finite λ-a.e., we show that

$$\int_\Omega \sum_{n=0}^{\infty} c_{\omega_n}\left|\epsilon(\omega_n)\right| d\lambda(\omega)$$

is finite. By Fubini's theorem and using the fact that $c_v\lambda(I_v) = 1$, we obtain

$$\int\limits_{\Omega} \sum_{n=0}^{\infty} c_{\omega_n} \left|\epsilon(\omega_n)\right| d\lambda(\omega) = \sum_{n=0}^{\infty} \int\limits_{\Omega} c_{\omega_n} \left|\epsilon(\omega_n)\right| d\lambda(\omega) = \sum_{n=0}^{\infty} \sum_{|v|=n} c_v \left|\epsilon(v)\right| \lambda(I_v)$$
$$= \sum_{n=0}^{\infty} \sum_{|v|=n} \left|\epsilon(v)\right| = \sum_{v \in T} \left|\epsilon(v)\right| < \infty. \tag{6}$$

Thus, we may define

$$F(\omega) = \sum_{j=0}^{\infty} c_{\omega_j} \epsilon(\omega_j) \quad \lambda\text{-a.e.},$$

which by (6) is in $L^1(\lambda)$.

Let $v \in T$ with $|v| = n$, and recall that for $j > n$, $D_j(v)$ is the set of descendants of v of length j. Using (3), we have

$$\begin{aligned}
\int\limits_{I_v} F(\omega)\, d\lambda(\omega) &= \sum_{j=0}^{\infty} \int\limits_{I_v} c_{\omega_j} \epsilon(\omega_j)\, d\lambda(\omega) \\
&= \sum_{j=0}^{n} \int\limits_{I_v} c_{v_j} \epsilon(v_j)\, d\lambda(\omega) + \sum_{j=n+1}^{\infty} \sum_{w \in D_j(v)} \int\limits_{I_w} c_w \epsilon(w)\, d\lambda(\omega) \\
&= \sum_{j=0}^{n} c_{v_j} \frac{\epsilon(v_j)}{c_v} + \sum_{j=n+1}^{\infty} \sum_{w \in D_j(v)} \epsilon(w) \\
&= \sum_{j=0}^{n} \frac{\epsilon(v_j)}{\prod_{k=j}^{n-1} q_{v_k}} = \mu(I_v).
\end{aligned}$$

Thus $d\mu$ extends to the complex measure $F d\lambda$ on the Borel sets of Ω, completing the proof of (a).

The proof of (b) is modeled on Exercises 6–8 in Section 3.3 of [1]. We shall, therefore, eliminate many of the details.

Let $f : T \to \mathbb{C}$ be a function such that $f(v) = 0$ for all $v \in T$, $|v| \leqslant 1$. For each vertex v with $|v| \geqslant 1$ let $\{f(w):\ w^- = v\}$ be the set of the q_vth roots of unity.

For each $n \in \mathbb{N}$, let R_n and S_n be the functions on Ω defined by

$$R_n(\omega) = f(\omega_n), \qquad S_n(\omega) = \sum_{j=1}^{n} \frac{R_j(\omega)}{j}.$$

Note that for any vertex $v \neq e$, R_n is constant on I_v if $|v| \geqslant n$, and

$$\sum_{u \in D_n(v)} f(u) = 0$$

since the sum of the qth roots of unity is 0 for all integers $q > 1$. Thus

$$\int\limits_{I_v} R_n\, d\lambda = 0 \quad \text{if } |v| < n. \tag{7}$$

This implies that if $j, k \in \mathbb{N}$, $j < k$, then

$$\int\limits_{\Omega} R_j \bar{R}_k \, d\lambda = \sum_{|v|=j} R_j \int\limits_{I_v} \bar{R}_k \, d\lambda = 0.$$

Using this, one can prove that the sequence $\{S_n\}$ is Cauchy in $L^2(\lambda)$. This implies that $\{S_n\}$ is convergent in $L^2(\lambda)$ (and hence also in $L^1(\lambda)$) to some function S. We claim that $\{S_n\}$ converges to S pointwise λ-a.e.

Let $N, k, n \in \mathbb{N}$, $N \leqslant k < n$, and let $v \in T$ with $|v| = k$. Then $S_N - S_k$ is constant on I_v, and we can show that

$$\int\limits_{I_v} |S_N - S_n|^2 \, d\lambda \geqslant \int\limits_{I_v} |S_N - S_k|^2 \, d\lambda.$$

Given $\delta > 0$, let

$$\begin{aligned} A_k = \bigl\{\omega \in \Omega\colon \ & \bigl|S_N(\omega) - S_j(\omega)\bigr| \leqslant \delta \text{ for all } j,\ N \leqslant j < k, \text{ and} \\ & \bigl|S_N(\omega) - S_k(\omega)\bigr| > \delta\bigr\} \end{aligned}$$

and define

$$A = \bigl\{\omega \in \Omega\colon \bigl|S_N(\omega) - S_k(\omega)\bigr| > \delta \text{ for some } k,\ N < k < n\bigr\}.$$

Note that $A = \bigcup_{k=N+1}^{n-1} A_k$, a disjoint union. Suppose $A_k \cap I_v \neq \emptyset$. Since $S_N - S_j$ is constant on I_v for all $j \leqslant k$, it follows that $I_v \subset A_k$. Thus A_k is the disjoint union of intervals generated by vertices of length k. We get

$$\lambda(A) \leqslant \frac{1}{\delta^2} \int\limits_{A} |S_N - S_n|^2 \, d\lambda \leqslant \frac{1}{\delta^2} \|S_N - S_n\|^2_{L^2(\lambda)}.$$

For each $M, N \in \mathbb{N}$, consider the set

$$A_{M,N} = \bigl\{\omega \in \Omega\colon \bigl|S_N(\omega) - S_k(\omega)\bigr| > 1/M \text{ for some } k > N\bigr\}.$$

Define

$$E = \bigcup_{M=1}^{\infty} \bigcap_{N=1}^{\infty} A_{M,N}.$$

If $\omega \in \Omega - E$, then for each $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $n_1, n_2 \in \mathbb{N}$ with $n_1 > n_2 > N$,

$$\bigl|S_{n_1}(\omega) - S_{n_2}(\omega)\bigr| \leqslant \bigl|S_{n_1}(\omega) - S_N(\omega)\bigr| + \bigl|S_{n_2}(\omega) - S_N(\omega)\bigr| < 2/M,$$

and so the sequence $\{S_n(\omega)\}$ converges. To complete the proof of the claim, it remains to show that $\lambda(E) = 0$. Since $\lambda(A_{M,N}) \leqslant M^2 \|S_N - S\|^2_{L^2(\lambda)}$, we obtain

$$\lambda\left(\bigcap_{N=1}^{\infty} A_{M,N}\right) \leqslant M^2 \lim_{N\to\infty} \|S_N - S\|^2_{L^2(\lambda)} = 0.$$

It follows that $\lambda(E) = 0$.

For each $v \in T$, let us define $\mu(I_v) = \int_{I_v} S\,d\lambda$. Then, using (7) and the fact that R_n is constant on I_v for $n \leqslant |v|$, we have

$$\mu(I_v) = \sum_{n=1}^{|v|} \frac{f(v_n)}{n} \frac{1}{c_v}.$$

Thus

$$\sum_{v \neq e} \left| \mu(I_v) - \frac{1}{q_{v^-}} \mu(I_{v^-}) \right| = \sum_{v \neq e} \frac{|f(v)|}{|v| c_v} = \sum_{n=1}^{\infty} \sum_{|v|=n} \frac{1}{n c_v}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{|v|=n} \lambda(I_v) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Therefore $\mu \notin \mathcal{B}_0$. This completes the proof of the proper inclusion $\mathcal{B}_0 \subsetneqq \mathcal{M}_A$. □

3. Extending distributions to complex measures

In this section we explore the conditions needed for a distribution on Ω to extend to a complex measure on the Borel sets of Ω.

Lemma 3.1. *Let $I_1, \ldots, I_N, J_1, J_2, \ldots,$ be intervals such that*

$$\bigcup_{n=1}^{N} I_n = \bigcup_{m=1}^{\infty} J_m. \tag{8}$$

Then there exists $M \in \mathbb{N}$ such that

$$\bigcup_{n=1}^{N} I_n = \bigcup_{m=1}^{M} J_m.$$

Proof. The set $\{J_n\}$ is an open cover of the left-hand side of (8), which is compact. Thus there exists a finite subcover. □

Lemma 3.2. *Assume*

$$S = \coprod_{n=1}^{\infty} I_n = \coprod_{m=1}^{\infty} J_m,$$

where the sets I_n and J_m are intervals. Then there exists a sequence of intervals $\{L_k\}_{k \in \mathbb{N}}$ such that

$$S = \coprod_{k=1}^{\infty} L_k$$

and such that each I_n and each J_m is a finite union of L_k's.

Proof. Well order the set of pairs $\{(n,m)\colon I_n \cap J_m \neq \emptyset\}$ and let the kth element be $L_k = I_n \cap J_m$ and use the notation $k \sim (n,m)$ for this correspondence. Since the intersection of two intervals is one or the other or is empty, L_k is either I_n or J_m. The sets L_k are pairwise disjoint and their union is S. If I_n is contained in some J_m, then $L_k = I_n$, where $k \sim (n,m)$. If I_n contains $J_{m_1}, \ldots, J_{m_t}$ (a finite set by Lemma 3.1), then

$$I_n = \coprod_{k_i \sim (n,m_i),\, 1 \leqslant i \leqslant t} L_{k_i}.$$

By symmetry, the same is true of the intervals J_m. □

Theorem 3.1. *Let $\{I_n\}$ and $\{J_m\}$ be sequences of pairwise disjoint intervals such that*

$$\coprod_{n=1}^{\infty} I_n = \coprod_{m=1}^{\infty} J_m.$$

If μ is an absolutely summable distribution, then

$$\sum_{n=1}^{\infty} \mu(I_n) = \sum_{m=1}^{\infty} \mu(J_m).$$

Proof. By Lemma 3.2, it suffices to assume that each interval I_n is a finite union of J_m's. By the absolute summability of μ, we may assume that the ordering is consistent with the unions, that is, that for all $N \in \mathbb{N}$ there exists $M_N \in \mathbb{N}$ such that

$$\coprod_{n=1}^{N} I_n = \coprod_{m=1}^{M_N} J_m.$$

Fix $\epsilon > 0$. Since $\sum_{n=1}^{\infty} \mu(I_n)$ and $\sum_{m=1}^{\infty} \mu(J_m)$ are absolutely summable, there exist $N' \in \mathbb{N}$ and $M' \in \mathbb{N}$, $M' \geqslant M_{N'}$, such that

$$\sum_{n>N'} \bigl|\mu(I_n)\bigr| < \epsilon/2 \quad \text{and} \quad \sum_{m>M'} \bigl|\mu(J_m)\bigr| < \epsilon/2.$$

Let $N \in \mathbb{N}$ be such that $N > N'$ and $M = M_N \geqslant M'$. Then

$$\coprod_{n=1}^{N} I_n = \coprod_{m=1}^{M} J_m,$$

$$\left|\sum_{n=1}^{\infty} \mu(I_n) - \sum_{n=1}^{N} \mu(I_n)\right| < \epsilon/2 \quad \text{and} \quad \left|\sum_{m=1}^{\infty} \mu(J_m) - \sum_{m=1}^{M} \mu(J_m)\right| < \epsilon/2.$$

Since $\sum_{n=1}^{N} \mu(I_n) = \sum_{m=1}^{M} \mu(J_m)$, we obtain

$$\left|\sum_{n=1}^{\infty} \mu(I_n) - \sum_{m=1}^{\infty} \mu(J_m)\right| < \epsilon,$$

proving the result. □

We now recall some basic definitions from measure theory.

Definition 3.1. A collection $\mathcal{A}$ of subsets of a set X containing $\emptyset$ and X is called an *algebra of sets* (respectively, σ*-algebra of sets*) if the union of any two (respectively, countably many) elements of $\mathcal{A}$ is also in $\mathcal{A}$ and the complement of any set in $\mathcal{A}$ is in $\mathcal{A}$.

Definition 3.2. A *measure on an algebra* is a nonnegative extended real-valued set function μ defined on an algebra of sets $\mathcal{A}$ such that $\mu(\emptyset) = 0$ and for any disjoint sequence $\{E_i\}$ of sets in $\mathcal{A}$ whose union is also in $\mathcal{A}$,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Proposition 3.1. *Any nonnegative distribution $\mu \in \mathcal{D}$ can be extended uniquely to a nonnegative finite measure on the Borel sets of Ω.*

Proof. Let $\mathcal{A}$ be the algebra generated by the intervals. Given any two intervals, either they are disjoint or one is contained in the other. Recalling Observation 1.1, we then get that $\mathcal{A}$ consists of the empty set and all finite disjoint unions of intervals. By Lemma 3.1, there are no infinite disjoint unions of (nonempty) intervals in $\mathcal{A}$. Thus, if $\mu \in \mathcal{D}$ is nonnegative, it is a finite measure on the algebra $\mathcal{A}$. By the Carathéodory extension theorem (cf. [9, p. 295]), μ can be extended to a measure on the smallest σ-algebra containing $\mathcal{A}$, i.e., the σ-algebra of the Borel sets of Ω. Furthermore, as $\mu(\Omega) < \infty$ such an extension is unique. □

Definition 3.3. Let μ be a distribution. Then define an extended real-valued function on intervals as follows:

$$|\mu|(I_v) = \sup \sum \left|\mu(I_{w_\alpha})\right|$$

taken over all partitions $\{I_{w_\alpha}\}$ of I_v. Observe that $|\mu|$ extends to a finitely additive extended real-valued function on finite unions of intervals. In particular, if $|\mu|(\Omega)$ is finite, then $|\mu|$ is a nonnegative distribution. We call $|\mu|$ the *total variation* of μ.

By Lemma 3.1 any partition $\{I_{w_\alpha}\}$ of I_v is finite. In particular, if $n = \max |w_\alpha|$, then $\sum |\mu(I_{w_\alpha})| \leqslant \sum_{u \in D_n(v)} |\mu(I_u)|$. Thus

$$|\mu|(I_v) = \lim_{n\to\infty} \sum_{u \in D_n(v)} \left|\mu(I_u)\right|.$$

In [10, Theorems 6.2 and 6.4] it is shown that if μ extends to a measure, then $|\mu|$ can be extended to a finite-valued measure. We shall prove a stronger version of this result in case μ is merely a distribution. Specifically, we show that a distribution μ is absolutely summable if and only if its total variation can be extended to a finite-valued measure, and this holds if and only if μ itself can be extended to a measure.

The following result is reminiscent of Lemma 6.3 of [10], but is more elementary.

Lemma 3.3. *For $N \in \mathbb{N}$ let $a_1, \ldots, a_N \in \mathbb{R}$ be such that*

$$\sum_{j=1}^{N} |a_j| \geqslant \left| \sum_{j=1}^{N} a_j \right| + 2.$$

Then for all $k \in \{1, \ldots, N\}$,

$$\sum_{j \neq k} |a_j| \geqslant 1.$$

Proof. Assume $\sum_{j \neq k} |a_j| < 1$. Then by two applications of the triangle inequality, we obtain

$$\sum_{j=1}^{N} |a_j| < 1 + |a_k| < 2 - \sum_{j \neq k} |a_j| + |a_k| \leqslant 2 + \left| \sum_{j=1}^{N} a_j \right|. \qquad \square$$

We are now ready to prove the main result of this section.

Theorem 3.2. *Let* $\mu \in \mathcal{D}$. *Then the following statements are equivalent*:

(a) $\mu \in \mathcal{M}$.
(b) μ *is absolutely summable.*
(c) $|\mu|(\Omega) < \infty$.

Proof. (a) $\Rightarrow$ (b) If $\mu \in \mathcal{M}$, then for any countable collection $\{I_n\}$ of nonoverlapping intervals the value of $\sum_n \mu(I_n)$ is independent of the ordering of the index set and thus the series $\sum_n \mu(I_n)$ must be absolutely convergent.

(b) $\Rightarrow$ (c) Assume $|\mu|(\Omega)$ is infinite. Then there exists an infinite sequence of vertices $\{v_0, v_1, \ldots\}$ with $v_0 = e$, $|v_{j+1}| > |v_j|$, and such that for each j, $|\mu|(I_{v_j}) = \infty$. Thus for each j, there exists $m_j \in \mathbb{N}$ such that

$$\sum_{u \in D_{m_j}(v_j)} \bigl|\mu(I_u)\bigr| \geqslant \bigl|\mu(I_{v_j})\bigr| + 2.$$

By Lemma 3.3, it follows that

$$\sum_{u \in D_{m_j}(v_j),\, u \neq v_{m_j}} \bigl|\mu(I_u)\bigr| \geqslant 1.$$

Define $A_j = \{u \in D_{m_j}(v_j)\colon u \neq v_{m_j}\}$. By passing to a subsequence we may assume without loss of generality that $|v_{j+1}| > m_j$ for every j. Thus $\{I_u\colon u \in A_j,\ j \in \mathbb{N}\}$ is a collection of nonoverlapping intervals. Setting $A = \bigcup_{j=1}^{\infty} A_j$, we obtain

$$\sum_{u \in A} \bigl|\mu(I_u)\bigr| = \sum_{j=1}^{\infty} \sum_{u \in A_j} \bigl|\mu(I_u)\bigr| = \infty.$$

Thus, μ is not absolutely summable.

(c) $\Rightarrow$ (a) If $|\mu|(\Omega)$ is finite, then $|\mu|$ is a nonnegative distribution such that $|\mu|(I_v) \geqslant |\mu(I_v)|$, for each $v \in T$. Thus $|\mu|$ and $\rho = \mu + |\mu|$ are both nonnegative distributions. By

Proposition 3.1, there exist unique measures $\widehat{|\mu|}$ and $\hat{\rho}$ on the Borel sets of Ω which are extensions of $|\mu|$ and ρ, respectively. Then $\hat{\rho} - \widehat{|\nu|}$ is a finite signed measure on the Borel sets of Ω which is an extension of μ. $\square$

4. Decomposition theorems

We now want to find out how far a distribution is from being a measure. Let v be a vertex such that $|\mu|(I_v) < \infty$. By considering the tree of descendants of v whose boundary is exactly I_v, it follows from Theorem 3.2 that μ may be extended to a measure.

Definition 4.1. Let $A \subset \Omega$ be a union of intervals. A distribution μ on A is called a *quasi-measure* if for every interval $I \subset A$, $\mu|I$ can be extended to a complex measure.

If μ is a real-valued quasi-measure on A, then B is a *positive set* (respectively, *negative set*) for μ if for every interval $I \subset A$, and measurable set $C \subset B \cap I$, $\mu(C) \geqslant 0$ (respectively, $\mu(C) \leqslant 0$).

A real-valued quasi-measure does not necessarily extend to a signed measure because there may be subsets with measure $+\infty$ and subsets with measure $-\infty$. On the other hand, a positive- (respectively, negative-) valued quasi-measure does extend to a positive (respectively, negative) measure.

Definition 4.2. (a) Let μ be a distribution and let ν be a positive distribution on Ω. We say that μ is *absolutely continuous* with respect to ν, and write $\mu \ll \nu$, if $\mu(I) = 0$ for every interval such that $\nu(I) = 0$.

(b) We say that a distribution μ is *concentrated* on $A \subset \Omega$ if $\mu(I) = 0$ whenever I is an interval such that $I \cap A = \emptyset$.

(c) We say that two distributions μ and ν are *mutually singular*, and write $\mu \perp \nu$, if there exist disjoint sets B and C such that μ is concentrated on B and ν is concentrated on C.

We now give a version of the Hahn, Jordan, and Lebesgue–Radon–Nikodym decomposition theorems for quasi-measures.

Theorem 4.1. *Let $A \subset \Omega$ be a union of intervals and let μ be a quasi-measure on A.*

(a) (Jordan decomposition) *If μ is real-valued, then there exist positive measures μ_+ and μ_- on A such that for any interval $I \subset A$, $\mu(I) = \mu_+(I) - \mu_-(I)$.*
(b) (Hahn decomposition) *If μ is real-valued, then there exist a positive set B and a negative set C for μ such that $B \cup C = A$.*
(c) (Lebesgue–Radon–Nikodym decomposition) *If ν is a positive quasi-measure on A, then there exists a unique pair (μ_a, μ_s) of distributions on A such that $\mu = \mu_a + \mu_s$, $\mu_a \ll \nu$, $\mu_s \perp \nu$. If μ is positive and finite, then so are μ_a and μ_s. Moreover, there exists a unique $h \in L^1(\nu)$ such that $\mu_a(E) = \int_E h\, d\nu$ for each finite union of intervals E in A.*

Proof. (a) The total variation of μ is a positive quasi-measure, hence can be extended to a finite positive measure on A. Similarly, $(|\mu| + \mu)/2$ and $(|\mu| - \mu)/2$ can be extended to a finite positive measures μ_+ and μ_- on A.

(b) Let C be the support of μ_-, and $B = A - C$.

(c) Since ν is a positive quasi-measure on A, we may assume it has been extended to a measure on A. Write $A = \coprod_{n=1}^{\infty} I_n$, where I_n are (pairwise disjoint) intervals. Let μ_n be the measure which extends $\mu|I_n$ on I_n and is 0 on $A - I_n$. Then $\mu = \sum_{n=1}^{\infty} \mu_n$, i.e., for any finite union of intervals E, $\mu(E) = \sum_{n=1}^{\infty} \mu_n(E)$. Clearly, the support of μ_n is in I_n. Then by the classical Lebesgue–Radon–Nikodym decomposition theorem, there exist measures $\mu_{n,a}$ and $\mu_{n,s}$ with support in I_n such that $\mu_n = \mu_{n,a} + \mu_{n,s}$, $\mu_{n,a} \ll \nu$, $\mu_{n,s} \perp \nu$, and there exists a unique $h_n \in L^1(\nu)$ with support in I_n such that $\mu_{n,a}(E) = \int_E h_n \, d\nu$ for each finite union of intervals in A. Then $\mu_a = \sum \mu_{n,a}$, $\mu_s = \sum \mu_{n,s}$ and $h = \sum h_n$ satisfy the conclusion of the theorem. If, moreover, μ is positive, then it can be extended to a measure and the result follows from the classical case. $\square$

Definition 4.3. Given a distribution μ, the *measure support of* μ is defined by the set

$$A_\mu = \bigcup_{\{v \in T:\ |\mu|(I_v) < \infty\}} I_v.$$

Clearly, if $A_\mu = \Omega$, then μ can be extended to a measure by Theorem 3.2.

Furthermore, the measure support of a distribution is the largest set on which these decomposition theorems hold, by Theorem 4.2 below.

In the following two examples, let T_3 denote a homogeneous tree of degree 3.

Example 4.1. Let μ be the distribution in Example 1.1 applied to T_3. Notice that for every vertex $v \notin \{w_k\colon k = 0, 1, \ldots\}$, $\mu(I_v) = 2^{-|v|+n}$, where n is the largest integer such that v is a descendant of v_n, or $\mu(I_v) = 0$ if v is not a descendant of v_1 or w_1. Since $\mu|I_v$ is positive, it is absolutely summable and so it can be extended to a positive measure. We thus can get a σ-finite measure on the noncompact space $\Omega - \{\omega_0\}$, where $[w_0, w_1, \ldots] = [e, \omega_0)$. It is σ-finite because it can be written as the countable union of set of finite measure, namely

$$\Omega - \{\omega_0\} = I_{u_1} \amalg \coprod_{n=1}^{\infty} I_{v_n},$$

where u_1 is the neighbor of e other than v_1 and w_1. If I is any interval containing ω_0, then $\mu|\Omega - I$ can be extended to a finite nonnegative measure. Thus the measure support of μ is $\Omega - \{\omega_0\}$.

The following is an example of a distribution on T_3 whose measure support is the empty set.

Example 4.2. Let $\mu(\Omega) = 0$, $\mu(I_{v_1}) = \mu(I_{u_1}) = 1$ and $\mu(I_{w_1}) = -2$, where, we recall, u_1, v_1, and w_1 are the neighbors of e. Now assume that v is a vertex with children u and w such that $\mu(I_v) \neq 0$. Then let $\mu(I_u) = -\operatorname{sign} \mu(I_v)$ and $\mu(I_w) = \mu(I_v) - \mu(I_u)$. Consequently, every vertex v has a path of descendants $[x_1, x_2, \ldots]$ such that $\mu(I_{x_n}) \to -\infty$ as well as one going to $+\infty$. In particular, $|\mu|(I_v) = \infty$ for every vertex v. Thus, the measure support of μ is the empty set.

In general, we cannot get a Jordan decomposition of a distribution which is not absolutely summable because if we could write $\mu = \mu_+ - \mu_-$ with μ_+, μ_- positive distributions, then μ_+ and μ_- could be extended to finite positive measures, so μ itself would be extendible to a finite signed measure, contradicting Theorem 3.2.

Theorem 4.2. *Let μ be a distribution which is not absolutely summable* (*so that $A_\mu \neq \Omega$*). *If I is an interval in $\Omega - A_\mu$, then for all positive integers n there exist finite disjoint unions of intervals J_n and K_n in I such that $\mu(J_n) > n$ and $\mu(K_n) < -n$.*

In particular, if μ is not an absolutely summable distribution, then there cannot be a Hahn decomposition on the complement of A_μ: $\Omega - A_\mu = B \cup C$, with B and C disjoint, μ positive on B and negative on C.

Proof. Assume that we cannot find the sets K_n satisfying the condition $\mu(K_n) < -n$. That is, suppose there exists a natural number N such that for all finite disjoint unions of intervals $K \subset I$, $\mu(K) \geqslant -N$. Since the complement (with respect to I) of a finite disjoint union of intervals is also a finite disjoint union of intervals, if K' is such a set, then $\mu(K') \leqslant \mu(I) + N$. So for each such set K, $-N \leqslant \mu(K) \leqslant \mu(I) + N$. Thus, there exists some positive number M such that $-M \leqslant \mu(K) \leqslant M$. Let $\{I_j\}$ be a finite set of disjoint intervals whose union is I. Let K be the union of the I_j with $\mu(I_j) \geqslant 0$ and let K' be the union of the I_j with $\mu(I_j) < 0$. Then

$$\sum \bigl|\mu(I_j)\bigr| = \mu(K) - \mu(K') \leqslant 2M.$$

Thus μ is of bounded variation on I, a contradiction.

Similarly replacing μ by $-\mu$, we see that the sets $J_n \subset I$ satisfying $\mu(J_n) > n$ must exist. □

5. Good ultrametric spaces

Definition 5.1. A metric d on a space X is said to be an *ultrametric* if for any points $x, y, z \in X$, $d(x, y) \leqslant \max\{d(x, z), d(y, z)\}$, or equivalently, two of the three values $d(x, y), d(x, z), d(y, z)$ are the same and the third is no larger.

Note that in any ultrametric space (X, d), if $x \in X$, $r > 0$, and

$$y \in B_r(x) = \bigl\{v \in X\colon\ d(v, x) \leqslant r\bigr\},$$

then $B_r(y) = B_r(x)$. Thus, if $z \in X$, $s > 0$, and $y \in B_r(x) \cap B_s(z)$, then $B_r(x) = B_r(y)$ and $B_s(z) = B_s(y)$. Hence, if two balls intersect, then the ball with the smaller radius is contained in the other.

Definition 5.2. An ultrametric d is said to be *good* (cf. [3]) if the set

$$D = \{d(x, y)\colon x, y \in X,\ x \neq y\}$$

is discrete, has 0 as a limit point, and if each ball $B_r(x)$ is compact.

Observation 5.1. *Every infinite compact ultrametric space is good.*

Proof. Assume X is an infinite compact ultrametric space. Trivially, it is locally compact. For $x \in X$ let $D_x = \{d(x, y)\colon x \neq y\}$ so that

$$D = \bigcup_{x \in X} D_x.$$

We first show that D_x is discrete. Assume $r \in D_x$ and there is a sequence $\{x_n\}$ in X such that $d(x_n, x) \neq r$ but $\lim_{n\to\infty} d(x_n, x) = r$. Since X is compact, we may assume that $\{x_n\}$ converges to some point x_0. Then

$$d(x, x_0) = \lim_{n\to\infty} d(x, x_n) = r.$$

Since $d(x, x_n) \neq r$, $d(x_n, x_0) = \max\{d(x, x_n), d(x, x_0)\} \geqslant r$, which contradicts the convergence. Thus, D_x is discrete.

For $x \in X$ and $r > 0$, let $D_x^r = D_x \cap [r, \infty)$, and set $D^r = D \cap [r, \infty) = \bigcup_{x \in X} D_x^r$. By the ultrametric property, if $y \in B_r(x)$, then $D_y^r = D_x^r$. For a fixed r there exist $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{k=1}^n B_r(x_k)$ and thus, $D^r = \bigcup_{k=1}^n D_{x_k}^r$ is discrete. This proves that D is discrete.

To show that D has 0 as a limit point, assume $\epsilon > 0$ is a lower bound of D. Let $\{x_n\}$ be an infinite sequence of distinct points of X. Then for $n \neq m$, $d(x_n, x_m) > \epsilon$, so $\{x_n\}$ has no convergent subsequences which contradicts the compactness of X. Thus, 0 must be a limit point of D. Since D is discrete, balls are both open and closed, hence compact. $\square$

Example 5.1. Let T be a rooted tree with boundary Ω such that no vertex has exactly two neighbors and let f be a positive function on T whose values along any path from the root to the boundary decrease to zero. If ω and ω' are distinct boundary points, let $d(\omega, \omega') = f(\omega \wedge \omega')$. Then (Ω, d) is a compact ultrametric space.

Example 5.2. Let T be a rooted tree with boundary Ω and a distinguished boundary point ω_0. Let f be a positive function on T with discrete range whose values along any path to any boundary point other than ω_0 decrease to zero and whose values along any path tending to ω_0 increase to ∞. If $\omega, \omega' \in \Omega - \{\omega_0\}$ are distinct, then the doubly infinite paths (ω, ω_0) and (ω', ω_0) have as their intersection an infinite path $[v, \omega_0)$. Define $\omega \vee \omega' = v$ and set $d(\omega, \omega') = f(\omega \vee \omega')$. Then $(\Omega - \{\omega_0\}, d)$ is a good ultrametric space which is not compact since distances can be arbitrarily large.

We now give a brief description of the main ideas of Choucroun's elegant paper [3] which yield the converse construction.

If (X, d) is a good ultrametric space, we can construct a tree T whose vertices v correspond to the balls B_v of X and whose edges are defined as follows. If v and v' are vertices, there is an edge between them if one of the balls B_v and $B_{v'}$ is strictly contained in the other, but no other ball fits properly between them. Given $x \in X$, let $\{d_n\}_{n\geqslant 0}$ be the decreasing sequence of elements of D_x. Let $v_n(x)$ be the vertex corresponding to $B_{d_n}(x)$ and let $v(x)$ be the boundary point corresponding to the path $[v_0(x), v_1(x), \ldots]$. The map $v : X \to \Omega$ is one-to-one, indeed, it is an isometry if we provide Ω with the ultrametric corresponding to the function $f(v) = \text{diameter}(B_v)$.

If X is compact, then X is itself a ball and the corresponding vertex serves as the root of the tree. Every boundary point yields a path from the root which then corresponds to a decreasing sequence of balls, and hence whose intersection is some point of X. Thus v is onto.

The equivalence of compact ultrametric spaces and boundaries of trees was also pointed out in [6] and [8], and in [7], for the case of homogeneous trees.

If X is not compact, then D is not bounded above and we can take a strictly increasing maximal sequence of balls whose union is all of X. The path of the corresponding vertices goes to a boundary point ω_0 which is independent of the sequence we have chosen. This is the only boundary point which is not in the image of v, and thus, X is in one-to-one correspondence with $\Omega - \{\omega_0\}$.

In both cases, having defined $f(v)$ to be the diameter of the ball B_v, the corresponding ultrametric on the boundary of the tree or the boundary with a point removed yields an isometry onto X.

Our aim is to translate all the results of this paper to a good ultrametric space X. We shall explain the translation below. The case of X compact follows precisely from what we have done in the earlier sections, since X may be identified with the boundary of a tree.

In order to understand the noncompact case, we need to study these questions on the noncompact ultrametric space $X = \Omega - \{\omega\}$. Let μ be a distribution on Ω. Then define a distribution $\tilde{\mu}$ on X by $\tilde{\mu}(I) = \mu(I) - \mu(\Omega)\lambda(I)$. $\tilde{\mu}$ is then a finitely additive function on the intervals not containing ω, i.e., on the balls of X. Conversely, let $\tilde{\mu}$ be a distribution on X. Let $I = I_{\omega_n}$ be an interval containing ω, and define

$$\mu(I) = -\sum_{|v|=n,\, v\neq\omega_n} \tilde{\mu}(I_v).$$

This yields a one-to-one correspondence between the distributions μ on Ω such that $\mu(\Omega) = 0$ and the distributions $\tilde{\mu}$ on X. Since any distribution ν on Ω is of the form $\mu + c\lambda$ where $\mu(\Omega) = 0$ and $\lambda \in B_0$, this does not affect any of our inclusions which will be proved to be valid on $\Omega - \{\omega\}$.

Thus we may identify a good ultrametric space X with Ω or $\Omega - \{\omega\}$ (depending on whether or not X is compact), where $\omega \in \Omega = \partial T$ for some tree T. The intervals $I_v \subset \Omega$ correspond to balls $B_v \subset X$. Just as each I_v is the disjoint union $\coprod_{w^- = v} I_w$, each ball B in X is the disjoint union of smaller balls none of which is contained in a larger ball strictly contained in B. Thus we get a natural definition of the distributions on X. Borel measures on X are taken with respect to the balls.

If B is a ball of a good ultrametric space, let $q(B)$ be the number of proper maximal sub-balls.

For the compact case, Lebesgue measure on X is well defined by letting $\lambda(X) = 1$, and if B' is a maximal proper sub-ball of a ball B, then $\lambda(B') = \lambda(B)/q(B)$. All the spaces $\mathcal{B}_0$, $\mathcal{M}_{\mathrm{AC}}$, $\mathcal{M}$, $\mathcal{D}_{\mathrm{AS}}$, $\mathcal{D}_{\mathrm{FTV}}$, $\mathcal{D}$ are easily defined in terms of the space X.

For the noncompact case, the Lebesgue measure on X requires fixing a ball $\hat{B}_0$. By the discreteness of D, there is a unique strictly increasing maximal sequence of balls $\{\hat{B}_n\}_{n\geqslant 0}$ which covers X. Let $q_n = q(\hat{B}_n)$ for $n > 0$ and $q_0 = q(\hat{B}_0) + 1$. Then define $\lambda(\hat{B}_n) = 1 - 1/q_0 q_1 \ldots q_n$ and otherwise extend as in the compact case. A distribution on X then is a finitely additive function on the balls. Using the notation B^- for the smallest ball strictly containing B, a distribution μ is in $\mathcal{B}_0$ if

$$\sum_{B \neq \hat{B}_n} \left| \mu(B) - \frac{1}{q(B^-)} \mu(B^-) \right| < \infty.$$

The space $\mathcal{M}_{\mathrm{AC}}$ is defined as the set of multiples of λ by an L^1-function with respect to λ, and $\mu \in \mathcal{D}_{\mathrm{AS}}$ if and only if $\sum \mu(B_n)$ converges absolutely for each sequence of pairwise disjoint balls.

Since any two choices of $\{\hat{B}_n\}_{n\geqslant 0}$ are eventually the same, the space $\mathcal{B}_0$ is well defined, and for any two choices of λ, each one is absolutely continuous with respect to the other. Thus all of the classes are well defined and so all the results in this paper now hold on an arbitrary good ultrametric space.

Similarly, a complex measure on X corresponds to a complex measure μ on Ω such that $\mu(\Omega) = 0$.

Notice that the absolutely summable measures on X correspond exactly to the absolutely summable measures on Ω since any disjoint union of intervals on Ω differs by at most one interval from a disjoint union of intervals in X. Furthermore, since X and Ω differ by one point, the corresponding L^1 spaces with respect to λ are the same.

Example 5.3. Let p be a prime. Every rational number r can be written uniquely as $r = (m/n)p^q$, where q, m, n are pairwise relatively prime integers, $n > 0$. Set $|r|_p = p^{-q}$. Then $d(r_1, r_2) = |r_1 - r_2|_p$ is an ultrametric on $\mathbb{Z}$ and $\mathbb{Q}$. The completions with respect to d yield the good ultrametric spaces $\mathbb{Z}_{(p)}$ and $\mathbb{Q}_{(p)}$, the former compact and the latter noncompact.

More generally, let R be a discrete valuation ring with valuation ν. If $d(r_1, r_2) = \nu(r_1 - r_2)$ for $r_1, r_2 \in R$ then d is an ultrametric, and the completion of R with respect to d is a good ultrametric space.

Thus all the results of this paper are valid on complete discrete valuation rings, including $\mathbb{Z}_{(p)}$ and $\mathbb{Q}_{(p)}$.

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