A Riesz decomposition theorem on harmonic spaces without positive potentials

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Abstract. In this paper, we give a new definition of the flux of a superharmonic function defined outside a compact set in a Brelot space without positive potentials. We also give a new notion of potential in a BS space (that is, a harmonic space without positive potentials containing the constants) which leads to a Riesz decomposition theorem for the class of superharmonic functions that have a harmonic minorant outside a compact set. Furthermore, we give a characterization of the local axiom of proportionality in terms of a global condition on the space.

1. Introduction

The Riesz decomposition theorem for positive superharmonic functions states that any positive superharmonic function on a region of hyperbolic type in $\mathbb{C}$ can be expressed uniquely as the sum of a nonnegative potential and a harmonic function. This result is of no interest when the region is parabolic because there are no nonconstant positive superharmonic functions.

By a Riesz decomposition theorem for a class of superharmonic functions we mean a unique representation of each member of this class as a global harmonic function plus a member of the class of a special type. For a harmonic space with a Green function, this special class is the class of nonnegative potentials. In this case the most natural class to consider is the admissible superharmonic functions, which are the superharmonic functions that have a harmonic minorant outside a compact set. Indeed, the admissible superharmonic functions are precisely the superharmonic functions which can be written uniquely as sums of a nonnegative potential plus a harmonic function. This follows from the fact that every admissible superharmonic function in a harmonic space with a Green function possesses a global harmonic minorant and hence, the function minus its greatest harmonic minorant is a nonnegative potential.

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In this article, we give a Riesz decomposition theorem for the class of the admissible superharmonic functions on a harmonic space without positive potentials. This class was first studied by Anandam in [2]. See also [6] for a reference on the admissible subharmonic functions.

In this paper, we give a new definition of the flux of a function that is superharmonic outside a compact set which is equivalent to the various definitions of flux in the works of Anandam (see [2] through [8]). It also encompasses the definitions of flux in the discrete setting given in [1], [9], and [15]. We then introduce a new class of potentials (called $H$-potentials) in the axiomatic setting that was first introduced in the recurrent tree and the complex plane settings in [15] and [16]. This new class allows us to obtain a global Riesz decomposition theorem for the class of superharmonic functions that have a harmonic minorant outside a compact set. We also give a characterization of the local axiom of proportionality in terms of a global condition on the space.

Many axiomatic treatments of potential theory have been formulated (for a survey and a historical context, see [13]). In this work we make use of the axiomatic theory of harmonic and superharmonic functions developed by Brelot (see [12]).

**Definition 1.1.** A **Brelot space** is a connected, locally connected, locally compact but not compact Hausdorff space $W$ together with a harmonic structure in the following sense. For each open set $U \subset W$ there is an associated real vector space of real–valued continuous functions on $U$ (which are called **harmonic functions on** $U$) satisfying the sheaf property, the regularity axiom, and the Harnack property.

The **harmonic support** of a superharmonic function $s$ is the complement of the largest open set on which $s$ is harmonic.

In a Brelot space, the Minimum Principle for superharmonic functions holds: A nonnegative superharmonic function on a domain $U$ in a Brelot space is either identically zero or positive everywhere on $U$ ([12], p. 71).

**Definition 1.2.** A superharmonic function $s$ on a Brelot space $W$ is said to be **admissible** if there are a compact set $K$ and a harmonic function $h$ on $W \setminus K$ such that $h(x) \leq s(x)$ for all $x \in W \setminus K$.

Clearly, positive superharmonic functions and superharmonic functions of compact harmonic support are admissible. We shall see in Proposition 2.3 that $K$ may be taken to be $K_0$, any fixed compact set independent of the superharmonic function.

Any nonnegative superharmonic function which has a harmonic minorant has a greatest harmonic minorant (see [12], p. 87).
**Definition 1.3.** A nonnegative superharmonic function on an open subset $U$ of a Brelot space is called a **positive potential** (or briefly, a **potential**) if its greatest harmonic minorant on $U$ is identically zero.

**Definition 1.4.** A **BH space** is a Brelot space whose sheaf of harmonic functions contains the constants. A **BP space** is a BH space on which there is a positive potential. A **BS space** (short for espace Brelot sans potentiel positif) is a BH space on which no positive potential exists.

Any open subset of $\mathbb{R}^n$ for $n \geq 3$ is a BP space, while the complement of any subset of $\mathbb{R}^2$ of logarithmic capacity zero is a BS space.

**Definition 1.5.** A BP space is said to satisfy the **axiom of proportionality** if any two potentials with the same one–point harmonic support are proportional.

**Theorem 1.1** ([17], p. 139). *In a Brelot space without potentials all positive superharmonic functions are harmonic and proportional. In particular, in a BS space, every positive superharmonic function must be constant.*

Thus, a Brelot space which possesses positive superharmonic functions which are not harmonic, has potentials.

The following result is the Riesz decomposition theorem for admissible superharmonic functions on a BP space.

**Theorem 1.2** ([6], p. 66). *In a BP space a superharmonic function is admissible if and only if it can be written uniquely as the sum of a potential and a function harmonic on the whole space.*

**Theorem 1.3** ([20], Theorem 16.1, [2], Theorem 3.6, and [14]). *If $\Omega$ is a Brelot space with positive potentials and a countable base of neighborhoods or if $\Omega$ is a BS space, then for any $x \in \Omega$, there exists a superharmonic function with harmonic support \{x\}.*

By Theorem 1.3 and Theorem 1.2, if $\Omega$ is a Brelot space with potentials and a countable base of neighborhoods, then for each $x \in \Omega$ there exists a potential with harmonic support \{x\}.

### 2. The flux of a superharmonic function

The concept of flux on a BS space was introduced for the purpose of associating a harmonic function on $\Omega$ to a function that is harmonic outside a compact set.

Nakai (see [22] or Theorem 1.20 of [2]) proved that if $h$ is a function defined on a BP space $\Omega$ and harmonic on the complement of a compact set $K$, then...
then $h = h_\Omega + b$ outside a compact set, where $h_\Omega$ is harmonic on $\Omega$ and $b$ is bounded. This is not true if $\Omega$ is a BS space, but we shall describe an obstruction, called the flux of $h$, so that when the flux of $h$ is zero, then $h$ is the sum of a global harmonic function and a bounded function.

The concept of flux of a function superharmonic outside a compact subset of a BS space appeared in [8] and [19].

**Proposition 2.1** ([3], Theorem 1.17). Let $\Omega$ be a BS space, $K \subset \Omega$ compact not locally polar. Then there exists a function $H \geq 0$ unbounded and harmonic off $K$. If $K$ is outer regular, then $H$ may be chosen so that it tends continuously to 0 on $\partial K$.

**Definition 2.1.** Let $\Omega$ be a BS space, $K \subset \Omega$ a nonempty compact set. A function $H$ harmonic off $K$ is called a standard for $\Omega$ associated with $K$ if the following is true: given any function $h$ which is harmonic off an arbitrary compact set, there exist a unique function $h_\Omega$ harmonic on the whole space and a unique real number $a$ such that $b = h - aH - h_\Omega$ is bounded off a compact set and $\liminf_{x \to \partial K} b(x) = 0$, where the lim inf is taken with respect to the Alexandrov one-point compactification of $\Omega$. This means that for any increasing exhaustion $\{C_n\}$ consisting of compact sets, and for any positive number $\varepsilon$ there exists $N \in \mathbb{N}$ such that for all integers $n \geq N$ there is a point $x \in \Omega \setminus C_n$ such that $b(x) < \varepsilon$, and for all $y \in \Omega \setminus C_n$, $b(y) > -\varepsilon$. Note that by the uniqueness of $a$, $H$ must be unbounded.

By Theorem 9.7 of [10], the function $H(z) = \log |z|$ is a standard for $\mathbb{C}$ associated with the closed unit disk. In this case, $b$ has an actual limit of 0 at infinity because a function which is harmonic and bounded outside a compact set in $\mathbb{C}$ has a limit at infinity. On the other hand, for $\Omega = \mathbb{C} \setminus \{2\}$, $H$ is still a standard for $\Omega$ associated with the closed unit disk, but a bounded harmonic function outside a compact set will have two limits as we approach the boundary of $\Omega$ in the extended plane (i.e. as $z \to 2$ and as $z \to \infty$).

In [8] (see note following Lemma 2) the following result is shown.

**Theorem 2.1.** Let $\Omega$ be a BS space and let $H$ be a nonnegative harmonic function defined outside an outer regular compact set $K$, not identically 0, and tending to 0 at the boundary of $K$. Then $H$ is a standard for $\Omega$.

In [11] (Theorem 4.2), we proved that the outer regularity of the compact set is unnecessary.

In the remainder of this section and in the next section, we shall fix a BS space $\Omega$, a nonempty compact subset $K_0$ which is not locally polar, and a standard $H$ associated with $K_0$ tending to 0 on $\partial K_0$, which for convenience we extend to be identically 0 in the interior of $K_0$. Observe that this extension is subharmonic on all of $\Omega$. 

The following is the key result which we shall use to calculate the flux of a superharmonic function outside a compact set.

**Proposition 2.2 ([8], Proposition 1).** Let $s$ be superharmonic outside some compact set. Then there exist a superharmonic function $s_\Omega$ on $\Omega$ and a constant $\beta$ such that $s = s_\Omega + \beta H$ outside a compact set.

**Remark 2.1.** Let $s$ be a function superharmonic outside a compact set. When it is clear from the context that the values of $s$ on any particular compact set are unimportant, we shall use Proposition 2.2 to change $s$ on a compact set so that it is defined globally and it is superharmonic outside $K_0$. In particular, whenever it is useful, we shall assume that $s$ is defined globally, is superharmonic outside $K_0$ and is lower bounded on $K_0$.

The decomposition in Proposition 2.2 is not unique because we may increase $\beta$ by subtracting an appropriate multiple of $-H$ from $s_\Omega$, since $-H$ is globally superharmonic. Thus, it would be interesting to know the following: let $\beta_0 = \inf \beta$ over all $\beta$ such that $s = s_\Omega + \beta H$ for some $s_\Omega$ superharmonic on $\Omega$. When is $\beta_0 > -\infty$? We shall discuss this question at the end of this section.

**Remark 2.2.** Let $s$ be an admissible superharmonic function, and let $h_1$ be a function which is harmonic outside a compact set $K$ and such that $h_1 \leq s$ outside $K$. By taking the restriction of $h_1$ to an outer regular compact set containing $K$, we may assume that $K$ is outer regular. We may extend $h_1$ to a continuous function on $\Omega$. Let $h_1$ be a lower bound of $s$ on $K$ and let $b_2$ be an upper bound of $h_1$ on $K$. Then $h = h_1 - |b_2 - h_1| \leq s$ on $\Omega$ and $h$ is harmonic outside $K$. Thus, when we say that $s$ has a harmonic minorant $h$ outside a compact set, we mean that $h \leq s$ globally and $h$ is harmonic outside a compact set.

**Remark 2.3 ([2], p. 133).** The difference of the greatest harmonic minorants $h_i$ of a superharmonic function outside a compact set $K_i$ ($i = 1, 2$) is bounded.

**Definition 2.2.** Let $s$ be superharmonic outside a compact set. Define

$$A_s = \{ x \in \mathbb{R} : \text{there exists } h_\Omega \text{ harmonic on } \Omega \text{ such that } s - x H \geq h_\Omega \}. $$

**Lemma 2.1.** If $s$ is superharmonic outside a compact set, then $A_s$ is bounded above. Furthermore, $A_s \neq \emptyset$ if and only if $s$ has a harmonic minorant outside a compact set.

**Proof.** By Proposition 2.2, there exist $\beta \in \mathbb{R}$ and $s_\Omega$ superharmonic on $\Omega$ such that $s = \beta H + s_\Omega$ outside a compact set. For $x \in A_s$, let $h_\Omega$ be harmonic...
on $\Omega$ such that $s - \alpha H \geq h_{\Omega}$. If $\alpha > \beta$, then $s_\Omega - h_{\Omega}$ is a lower bounded superharmonic function (since it is necessarily bounded below on any compact set), hence is constant. But by Proposition 2.1, a constant cannot be bounded below by a positive multiple of $H$. Thus, $\alpha \leq \beta$ and so $A_s$ is bounded above by $\beta$.

Assume $\alpha \in A_s$. Then there exists $h_{\Omega}$ harmonic on $\Omega$ such that $s \geq \alpha H + h_{\Omega}$. Thus $s$ has a harmonic minorant outside a compact set. Conversely, if $s$ has a harmonic minorant $h$ outside a compact set, then $h = \alpha H + h_{\Omega} + b$ outside a compact set, for some $\alpha \in \mathbf{R}$, $h_{\Omega}$ harmonic on $\Omega$, and $b$ bounded. Then outside a compact set $s - \alpha H \geq h_{\Omega} + \inf b$, a global harmonic function. Thus $\alpha \in A_s$.

**Definition 2.3.** Let $s$ be a function on $\Omega$ superharmonic outside a compact set. Define the flux of $s$ at infinity (or simply the flux of $s$) with respect to $H$ by

$$\text{flux}(s) = \sup A_s.$$ 

By convention, $\text{flux}(s) = -\infty$ if $A_s = \emptyset$. By Lemma 2.1, the flux of $s$ is finite if $A_s$ is nonempty.

The following result ties together our definition of flux with various earlier definitions, as well as compiling many useful properties of flux.

**Theorem 2.2.** (a) If $h$ is harmonic outside a compact set, then the flux of $h$ is the unique constant $\alpha$ of Definition 2.1 such that $h = \alpha H + h_{\Omega} + b$ outside a compact set with $h_{\Omega}$ harmonic on $\Omega$ and $b$ bounded. In particular, $A_h = (-\infty, \alpha]$.

(b) If $h$ is bounded harmonic outside a compact set or harmonic everywhere, then the flux of $h$ is zero. If $s$ is superharmonic everywhere, then $\text{flux}(s) \leq 0$.

(c) Let $s$ be an admissible superharmonic function on $\Omega$. If $h_1$ and $h_2$ are the greatest harmonic minorants of $s$ outside compact sets $K_1$ and $K_2$, respectively, then the flux of $h_1$ and the flux of $h_2$ are equal.

(d) If $s$ is a function superharmonic outside a compact set $K$ and has a subharmonic minorant on $\Omega \setminus K$ (in particular, if $s$ is admissible), then its flux is equal to the flux of its greatest harmonic minorant on $\Omega \setminus K$. Consequently, admissible superharmonic functions have finite flux.

(e) The flux of a nonadmissible superharmonic function $s$ is equal to $-\infty$.

(f) Let $s$ be superharmonic outside a compact set, and write $s = s_{\Omega} + \beta H$ as in Proposition 2.2. Then $\text{flux}(s) = \text{flux}(s_{\Omega}) + \beta$.

(g) The set of functions which are superharmonic outside a compact set is closed under addition and scalar multiplication by a positive number and the flux is linear on that set.
If \( s \) is superharmonic outside a compact set and \( A_s \not= \emptyset \), then \( \text{flux}(s) \in A_s \), so that \( A_s = (-\infty, \text{flux}(s)] \).

If \( s \) is superharmonic on \( \Omega \) and \( \text{flux}(s) = 0 \), then \( s \) is harmonic on \( \Omega \).

**Proof.** We use Proposition 2.2, Lemma 2.1, Remark 2.3, and the following immediate facts:

(i) There are no nonconstant positive superharmonic functions, hence no nonharmonic superharmonic function can be bounded below on \( \Omega \) by a harmonic function.

(ii) If \( s \) is superharmonic outside a compact set, then for all \( a \in \mathbb{R} \) and \( c > 0 \), \( A_{s+a} = A_s + a \), and \( A_{cs} = cA_s \).

(iii) (Monotonicity of the flux) If \( s_1 \leq s_2 \), with \( s_1, s_2 \) superharmonic outside a compact set, then \( A_{s_1} \subseteq A_{s_2} \), so that \( \text{flux}(s_1) \leq \text{flux}(s_2) \).

(iv) If \( h_\Omega \) is harmonic on \( \Omega \) and \( h_\Omega + \gamma H \) is bounded below, then \( \gamma \geq 0 \).

(v) If \( s_1 \) and \( s_2 \) are superharmonic outside a compact set \( K \), then the greatest harmonic minorant of \( s_1 + s_2 \) is the sum of the greatest harmonic minorants of \( s_1 \) and \( s_2 \) on \( K \).

Part (a) follows from (iv). Part (b) follows from (a) and Lemma 2.1. Part (c) holds since Remark 2.3 implies that \( A_{h_1} = A_{h_2} \).

To prove (d), let \( h \) be the greatest harmonic minorant of \( s \) on \( \Omega \setminus K \). By (c), without loss of generality we may assume that \( K \) contains \( K_0 \). By (a), \( A_h \not= \emptyset \), and by (iii), \( A_h \subseteq A_s \), so \( A_s \not= \emptyset \). Let \( \alpha \in A_s \) so that \( s - \alpha H \geq h_\Omega \), a function harmonic on \( \Omega \). Thus \( s \geq \alpha H + h_\Omega \), which is harmonic off \( K_0 \). Then on \( \Omega \setminus K \), \( s \geq h \geq \alpha H + h_\Omega \), so by (iii) and (a), \( \text{flux}(s) \geq \text{flux}(h) \geq \alpha \). Since this is true for all \( \alpha \in A_s \), it follows that \( \text{flux}(s) = \text{flux}(h) \).

To prove (e), assume there exists \( \alpha \in A_s \). Then \( s \geq \alpha H + h_\Omega \), where \( h_\Omega \) is a function harmonic on \( \Omega \). Since \( \alpha H + h_\Omega \) is harmonic outside \( K_0 \), \( s \) is admissible.

Part (f) follows from (ii). To prove (g), let \( s_1, s_2 \) be superharmonic outside the same compact set \( K \), and let \( h_1 \) and \( h_2 \) be their respective greatest harmonic minorants outside \( K \). Then by (v), \( h_1 + h_2 \) is the greatest harmonic minorant of \( s_1 + s_2 \) outside \( K \). Thus by (d) and (a), \( \text{flux}(s_1 + s_2) = \text{flux}(h_1 + h_2) = \text{flux}(h_1) + \text{flux}(h_2) = \text{flux}(s_1) + \text{flux}(s_2) \). Linearity with respect to multiplication by a positive constant follows from (ii).

Part (h) follows from (d) and (a). Part (j) follows from (h) and (i). \( \square \)

**Remark 2.4.** The original definitions of flux given by Anandam [8] separately first for harmonic functions outside a compact set, then for global superharmonic functions, and finally for functions superharmonic outside a compact set, are equivalent to ours, by Theorem 2.2, parts (a), (d), (e) and (f).
Proposition 2.3. A superharmonic function on a BS space $\Omega$ is admissible if and only if it has a minorant which is harmonic outside $K_0$.

Proof. Let $s$ be an admissible superharmonic function on $\Omega$ with flux $\alpha$. Then by part (h) of Theorem 2.2, there exists a harmonic function $h_\Omega$ on $\Omega$ such that $s \geq \alpha H + h_\Omega$, which is harmonic outside $K_0$. The converse is obvious. □

We now respond to the question raised after Remark 2.1.

Proposition 2.4. Let $s$ be superharmonic outside a compact set in a BS space $\Omega$. Let $B$ be the set consisting of all $\beta \in \mathbb{R}$ such that $s = s_\Omega + \beta H$ outside a compact set as in Proposition 2.2. Then

(a) The set $B$ is an interval unbounded above and $\inf B \geq \text{flux}(s)$. In particular, if $s$ has finite flux, then $B$ is bounded below.

(b) When $\Omega = \mathbb{R}$ with the harmonic structure inherited from the Laplace operator, $s$ has finite flux if and only if $B$ is bounded below.

Proof. To prove (a) assume $\beta \in B$ and $\gamma > \beta$. If $s = s_\Omega + \beta H$ outside a compact set $K$, then $s = s'_\Omega + \gamma H$ outside $K$, where $s'_\Omega = s_\Omega - (\gamma - \beta)H$ which is globally superharmonic. Thus $\gamma \in B$, proving that $B$ is an interval unbounded above. On the other hand, since the flux of a superharmonic function on $\Omega$ is less than or equal to 0, if $s = s_\Omega + \beta H$ outside a compact set then $\beta \geq \text{flux}(s)$. Thus, if $\text{flux}(s)$ is finite, $B$ is bounded below.

By part (a), to prove (b) we need to show that if $B$ is bounded below then $s$ has finite flux. Let $K_0 = \{0\}$. The function $H(x) = |x|$ is a standard for $K_0$. Let us consider $s(x) = |x| - x^2$ for $x \in \mathbb{R}$, which has flux $-\infty$ since it does not have a harmonic minorant outside a compact set. Since $s$ is smooth on $\mathbb{R}\setminus\{0\}$ and its Laplacian is $-2$ there, $s$ is superharmonic on $\mathbb{R}\setminus\{0\}$. For $n \in \mathbb{N}$, let $\tilde{s}_n(x) = s(x) + (2n - 1)H(x)$ and observe that for $x \geq n$, $s'(x) \leq 0$ and for $x \leq -n$, $s'(x) \geq 0$, where $s'$ denotes the derivative of $s$. Thus the function

$$s_n(x) = \begin{cases} \tilde{s}_n(n) & \text{for } |x| \leq n \\ \tilde{s}_n(x) & \text{for } |x| \geq n \end{cases}$$

is globally superharmonic and $s(x) = s_n(x) + \beta_n H(x)$ outside $[-n,n]$, where $\beta_n = -(2n - 1)$. Thus, $B$ is unbounded below. □

3. Potentials in a BS space and Riesz decomposition of admissible superharmonic functions

In this section we shall present several classes of admissible superharmonic functions which in a BS space play the role analogous to that of positive
potentials in a BP space. We first introduce two operators which we use to define these classes.

**Definition 3.1.** Let $s$ be superharmonic on $\Omega \setminus K_0$. Let $E = \{U_n\}$ be an increasing exhaustion consisting of relatively compact regular sets containing $K_0$ (which exists by [21]). Let $h_n = h^U_n$, the solution of the Dirichlet problem with boundary values $s$ on $\partial U_n$. Define

$$D_\varepsilon s(x) = \lim_{n \to \infty} h_n(x)$$

if this limit exists locally uniformly, in which case $D_\varepsilon s$ is harmonic on $\Omega$.

**Definition 3.2.** Let $K_0$ be outer regular and let $s$ be superharmonic on $\Omega \setminus K_0$. Define $D_s$ to be the greatest harmonic minorant of $s$ on each component of the complement of $K_0$ if such minorant exists and $-\infty$ if it does not exist on that component.

We now present different classes of potentials introduced by Anandam in [6] and [7].

**Definition 3.3.** An admissible superharmonic function $s$ is said to be in the class $\mathcal{P}$ if there exists an exhaustion $E$ such that $D_\varepsilon(s - \alpha H)$ exists and is constant, where $\alpha$ is the flux of $s$. If, furthermore, that constant is 0 for some exhaustion $E$, $s$ is called a BS potential. Define the class $\mathcal{Q}$ as the collection of all admissible superharmonic functions $s$ satisfying the property: there exists $s' \in \mathcal{P}$ such that the difference of the greatest harmonic minorants of $s$ and $s'$ outside a compact set is bounded. This class is independent of the choice of the compact set.

**Observation 3.1.** Suppose $s$ is in class $\mathcal{P}$ and has flux $\alpha$, so that for some exhaustion $E$, $D_\varepsilon(s - \alpha H)$ exists and is constant. By Lemma 2, p. 235 in [4], $s - \alpha H$ is lower bounded. If $D_\varepsilon'(s - \alpha H)$ exists for some other exhaustion $E'$, then $D_\varepsilon'(s - \alpha H)$ is a lower bounded harmonic function, hence it is also constant.

Anandam proved the following partial Riesz decomposition theorem for admissible superharmonic functions on a BS space.

**Proposition 3.1 ([5], Lemmas 2 and 3).** Any admissible superharmonic function $s$ on a BS space is a sum of a function in the class $\mathcal{Q}$ and a harmonic function. This decomposition is unique up to an additive constant. If $s$ has compact harmonic support, then the element of $\mathcal{Q}$ can be chosen uniquely to be a BS potential.

One difficulty in working with these classes of potentials is that given an admissible superharmonic function $s$, there is no procedure for determining
whether $s$ is in such classes. To overcome this difficulty and the lack of uniqueness in the decomposition of Proposition 3.1, we introduce a new class of potentials.

**Definition 3.4.** An admissible superharmonic function $s$ is an $H$-potential if

$$
\liminf_{x \to \infty} \{D_s(x) - z H(x)\} = 0,
$$

where $D$ is the operator of Definition 3.2, $z$ is the flux of $s$, and the $\liminf$ is taken with respect to the Alexandrov one-point compactification of $\Omega$.

The following theorem shows that the $H$-potentials are the class of potentials best suited to describe all admissible superharmonic functions.

**Theorem 3.1 (Global Riesz Decomposition Theorem).** In a BS space $\Omega$ every admissible superharmonic function can be written uniquely as the sum of an $H$-potential and a harmonic function.

**Proof.** Let $s$ be an admissible superharmonic function. Since $D_s$ is the greatest harmonic minorant of $s$ outside $K_0$, by Theorem 2.2(a) and (d), there exist $h_\Omega$ harmonic and $b$ bounded such that $D_s = z H + h_\Omega + b$ outside a compact set, where $z$ is the flux of $s$. By adding the condition that

$$
\liminf_{x \to \infty} b(x) = 0,
$$

we get uniqueness in the above decomposition. Then the function $p = s - h_\Omega$ is admissible superharmonic with flux $z$ and $Dp = Ds - h_\Omega = z H + b$. By (1), $Dp - z H$ has inferior limit 0 at infinity, thus $p$ is an $H$-potential and $s = p + h_\Omega$, proving the existence of the decomposition.

To prove the uniqueness, assume $p_1 + h_1 = p_2 + h_2$ where $p_1$ and $p_2$ are $H$-potentials and $h_1$ and $h_2$ are harmonic on $\Omega$. Then $p_1 - p_2 = h_2 - h_1$, which is globally harmonic. In particular, $p_1$ and $p_2$ have the same flux $z$. By definition of $H$-potential, $Dp_j = z H + b_j$ where $b_j$ ($j = 1, 2$) is bounded and harmonic outside a compact set with $\liminf 0$ at infinity. Since $h_2 - h_1 = p_1 - p_2 = D(p_1 - p_2) = Dp_1 - Dp_2 = b_1 - b_2$, the function $b_1 - b_2$ can be extended to a global bounded harmonic function. Thus, $b_1 - b_2$ is constant. Since $\liminf b_1 = \liminf b_2 = 0$, $b_1 = b_2$ and hence $h_1 = h_2$.

**Corollary 3.1.** If $s$ is an $H$-potential with flux $z$, then $D_s - z H$ is bounded.

**Proof.** By the proof of Theorem 3.1, for any admissible superharmonic function $s$, the unique global harmonic function $h_\Omega$ in the decomposition of $D_s$ as $z H + h_\Omega + b$ (with $b$ bounded) is the same global harmonic function in the
decomposition of \( s \) as an \( H \)-potential plus a global harmonic function. Thus if \( s \) is itself an \( H \)-potential, then \( h_\Omega = 0 \), and so \( Ds = zH + b \).

In [15], where we restricted our attention to recurrent trees, we defined an \( H \)-potential to be an admissible superharmonic function \( s \) for which \( Ds = zH \), where \( z = \text{flux}(s) \) and \( K_0 \) is taken to be a single point. It can be easily shown that in this setting the two definitions of \( H \)-potential are equivalent.

4. Proportionality in BS spaces

**Definition 4.1.** A Brelot space \( \Omega \) satisfies the **local axiom of proportionality** if for each \( x \in \Omega \) and each relatively compact neighborhood \( U \) of \( x \), all potentials on \( U \) with harmonic support at \( x \) are proportional.

**Theorem 4.1 (Local Riesz Decomposition Theorem).** If \( \Omega \) is a BS space satisfying the local axiom of proportionality, then the following properties hold.

(a) For all \( x \in \Omega \) there exists a unique BS potential \( p_x \) with harmonic support \( \{x\} \) such that \( \text{flux}(p_x) = -1 \).

(b) [18] For all \( s \) superharmonic on \( \Omega \) there exists a unique Radon measure \( \mu \geq 0 \) such that for each relatively compact domain \( U \subset \Omega \) and for all \( x \in U \)

\[
\begin{align*}
    s(x) &= \int_U p_y(x)d\mu(y) + h(x),
\end{align*}
\]

where \( h \) is harmonic on \( U \) and \( p_y \) is the unique BS potential with harmonic support \( y \) and flux equal to \(-1\).

**Proof.** (a) By Theorem 1.3, given any \( x \in \Omega \), there exists a superharmonic function \( s_x \) with harmonic support \( \{x\} \). By Proposition 3.1, there exists a BS potential \( p \) with harmonic support \( \{x\} \). Since the flux of an admissible superharmonic function which is not harmonic is negative, the function \( p_x = -\frac{p}{\text{flux}(p)} \) is superharmonic, and hence, a BS potential with harmonic support \( \{x\} \) and flux \(-1\), proving the existence.

For the uniqueness, assume that \( p \) and \( q \) are BS potentials on \( \Omega \) having harmonic support \( \{x\} \) and flux \(-1\). Let \( U \) be a relatively compact regular neighborhood of \( x \). Since \( p \) and \( q \) are superharmonic on \( U \) and \( U \) is a BP space, there exist potentials \( p_1 \) and \( p_2 \) on \( U \) with harmonic support \( \{x\} \), and \( h_1, h_2 \) harmonic on \( U \), such that \( p = p_1 + h_1 \), and \( q = p_2 + h_2 \). By the local axiom of proportionality, there exists \( \lambda > 0 \) such that \( p_1 = \lambda p_2 \). Thus \( h = p - \lambda q \) is harmonic on \( U \), but it is also harmonic off \( x \), and therefore is harmonic on \( \Omega \). Thus

\[
-1 = \text{flux}(p) = \text{flux}(h + \lambda q) = \lambda,
\]
so \( \lambda = 1 \). This shows that \( p - q \) is harmonic on \( \Omega \). By Observation 3.1 we deduce that \( p - q \) is constant. Since \( p \) and \( q \) are BS-potentials, that constant must be 0. Thus \( p = q \) on \( \Omega \).

The following result is a superharmonic extension theorem for BH spaces. It was proved by Hervé [20] in the BP case and later by Anandam (Theorem 3.4 of [2]) in the BS case.

**Theorem 4.2.** Let \( \Omega \) be a BH space and let \( U \) be an open subset of \( \Omega \). If \( s \) is a superharmonic function on \( U \) with compact harmonic support \( K \), then there exists a superharmonic function \( s_\Omega \) on \( \Omega \) with harmonic support on \( K \) such that \( s_\Omega - s \) is harmonic in a neighborhood of \( K \).

The following is a new characterization of the local axiom of proportionality.

**Theorem 4.3.** In a BH space, the following statements are equivalent.
(a) The local axiom of proportionality holds.
(b) For any two superharmonic functions with the same one-point harmonic support, some nonzero linear combination of them is harmonic.

**Proof.** Assume that the local axiom of proportionality holds on \( \Omega \), and let \( s_1, s_2 \) be superharmonic on \( \Omega \) with harmonic support at \( x \in \Omega \). Let \( U \) be a relatively compact neighborhood of \( x \). Then \( s_1|U \) and \( s_2|U \) are superharmonic on \( U \) with harmonic support at \( x \), and being lower-semicontinuous, they are bounded below on a relatively compact set, so they have harmonic minorants. Let \( h_1 \) and \( h_2 \) be the greatest harmonic minorants of \( s_1|U \) and \( s_2|U \), respectively. Thus \( s_1|U - h_1 \) and \( s_2|U - h_2 \) are potentials on \( U \) with harmonic support at \( x \), and so for some \( a > 0 \), \( s_1|U - h_1 = a s_2|U - h_2 \). Hence \( (s_1 - a s_2)|U = h_1 - a h_2 \) which is harmonic on \( U \). But \( s_1 - a s_2 \) is harmonic outside \( x \), thus \( s_1 - a s_2 \) is harmonic on \( \Omega \), proving that (b) holds.

Conversely, suppose (b) holds. Let \( x \in \Omega \), \( U \) a relatively compact neighborhood of \( x \), and let \( p_1 \) and \( p_2 \) be potentials on \( U \) with harmonic support at \( x \). By Theorem 4.2, there exist superharmonic functions \( s_1 \) and \( s_2 \) on \( \Omega \) with harmonic support at \( x \) such that \( s_1 - p_1 \) and \( s_2 - p_2 \) are harmonic on a neighborhood of \( x \). Thus, there exist nonzero \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha s_1 + \beta s_2 \) is harmonic on \( \Omega \). Then \( \alpha (s_1 - p_1) + \beta (s_2 - p_2) \) is harmonic on \( U \), so \( \alpha p_1 + \beta p_2 \) is harmonic on \( U \). Letting \( \lambda = -\frac{\beta}{\alpha} \), we have that \( p_1 - \lambda p_2 \) is a harmonic function \( h \) on \( U \). Notice that \( \lambda \) cannot be negative since otherwise \( \lambda p_2 \) would be a potential on \( U \) and the sum of two potentials cannot be harmonic. Thus, \( \lambda > 0 \) and \( p_1 = \lambda p_2 + h \). But the greatest harmonic minorant of \( p_1 \) and of \( p_2 \) is zero. Hence \( h = 0 \) and so \( p_1 = \lambda p_2 \) on \( U \), proving the local axiom of proportionality. \( \square \)
Example 4.1. Let Ω be an open subset of \( R^n \) whose harmonic structure is defined by the Laplace operator \( \Delta \). Then Ω satisfies part (b) of Theorem 4.3, hence satisfies the local axiom of proportionality, since if \( s_1 \) and \( s_2 \) are superharmonic functions on Ω with support at \( x \in \Omega \), then \( \Delta s_2(x)s_1 - \Delta s_1(x)s_2 \) is harmonic on Ω.

References


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