Let \((X, \mathcal{M}, \mu)\) be a measure space which we shall assume in this section to be \(\sigma\)-finite.

The theory of measurable functions and the theory of integration is very similar to the theory for the Lebesgue measure.

So in this section we merely note down the relevant definitions and state the results which are true. The proofs are typically the same as the analogous results with the Lebesgue measure.
Measurable functions

Let \( f : X \to \mathbb{R} \cup \{-\infty, \infty\} \) be an extended real-valued function on \( X \). We say that \( f \) is measurable if the following is true:

\[
f^{-1}([-\infty, a)) = \{ x \in X : -\infty \leq f(x) < a \} \in \mathcal{M} \text{ for all } a \in \mathbb{R}.
\]

Using the fact that \( \mathcal{M} \) is a \( \sigma \)-algebra, we can prove that the measurability of \( f \) is equivalent to any of the following:

\[
f \text{ measurable} \iff f^{-1}([-\infty, a]) \in \mathcal{M} \text{ for all } a \in \mathbb{R}
\]

\[
\iff f^{-1}([a, \infty]) \in \mathcal{M} \text{ for all } a \in \mathbb{R}
\]

\[
\iff f^{-1}((a, \infty]) \in \mathcal{M} \text{ for all } a \in \mathbb{R}
\]

The set of measurable functions is closed under the basic arithmetic operations (sums, multiplications, divisions, powers, etc.)

The set of measurable functions is closed, as before, under the operations of taking the pointwise supremum or infimum of a countable collection of measurable functions.

Thus it is closed under taking the pointwise lim sup or lim inf, of a sequence of measurable functions, and therefore the pointwise limit of a sequence of measurable functions (if it exists).
We say that two extended real-valued functions \( f \) and \( g \) are equal \( \mu \)-a.e. if
\[
\{ x \in X : f(x) \neq g(x) \} \in \mathcal{M}, \text{ and it has } \mu\text{-measure 0.}
\]

It is true that if \( f \) is measurable and \( g = f \ \mu\)-a.e., then also \( g \) is measurable.

A function \( g : X \to \mathbb{R} \) is called simple if there exist finitely many sets \( E_k \in \mathcal{M}, \ k = 1, \ldots, N \), each \( E_k \) of finite \( \mu \)-measure, and there exist real numbers \( a_1, \ldots, a_N \) such that
\[
g = \sum_{k=1}^{N} a_k \chi_{E_k}.
\]

The approximation theorems we proved hold here as well:

**Proposition**

- Let \( f \) be a non-negative \( \mu \)-measurable function. Then there exists a sequence of simple functions \( \phi_k \) which increase pointwise with \( k \), and converge pointwise to \( f \).

- If \( f \) is \( \mu \)-measurable, then there exists a sequence of simple functions \( \phi_k \) such that \( \phi_k \) converges pointwise to \( f \) and \(|\phi_k|\) increases pointwise with \( k \).

- (Egorov’s theorem) Let \( \{f_k\}_k \) be a sequence of \( \mu \)-measurable functions defined on a set \( E \in \mathcal{M} \) with \( \mu(E) < \infty \) for which \( f_k \) converges pointwise \( \mu \)-a.e. to \( f \). Let \( \varepsilon > 0 \). Then there is a subset \( A_\varepsilon \) of \( E \) such that \( \mu(E \setminus A_\varepsilon) < \varepsilon \) and \( f_k \) converges uniformly on \( A_\varepsilon \) to \( f \).
The four-step procedure for defining the integral with respect to the measure $\mu$ of a $\mu$-measurable function $f$ works the same way here. It is denoted by $\int f(x) \, d\mu(x)$, or just $\int f \, d\mu$.

In particular, for the simple function $f(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x)$, the integral is well-defined by the formula

$$\int f \, d\mu = \sum_{k=1}^{N} a_k \mu(E_k).$$

We say that $f$ is $\mu$-integrable if $f$ is $\mu$-measurable and $\int |f| \, d\mu < \infty$.

All of the basic properties of the integral, such as linearity, monotonicity, additivity, etc are true in this setting. It is also true that

If $f \geq 0$, then $\int f \, d\mu = 0 \iff f = 0 \, \mu-\text{a.e.}$
The integral limit theorems also hold with respect to \( \mu \)-integration:

**Proposition**

1. **(Fatou's lemma)** Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of non-negative measurable functions. Then
\[
\liminf_{n \to \infty} \int f_n(x) \, d\mu(x) \geq \int \liminf_{n \to \infty} f_n(x) \, d\mu(x).
\]

2. **(Monotone convergence theorem)** Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of non-negative measurable functions that increases pointwise. Then
\[
\lim_{n \to \infty} \int f_n(x) \, d\mu(x) = \int \lim_{n \to \infty} f_n(x) \, d\mu(x).
\]

3. **(Dominated Convergence theorem)** Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of measurable functions converging pointwise \( \mu \)-a.e. to \( f \). Suppose there exists an integrable function \( g \) such that \( |f_n(x)| \leq g(x) \) for \( \mu \)-a.e. \( x \). Then
\[
\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu.
\]
The space $L^1(X, \mu)$

- The relation on the set of $\mu$-integrable functions which relates any two functions which agree $\mu$-a.e. is an equivalence relation.

- Any two functions in the same equivalence class have the same integral. And so we define $L^1(X, \mu)$ to be the set of equivalence classes with respect to this relation, where the integral of a given equivalence class is the integral of any function in that equivalence class.

- It is perhaps simpler to just view an element of $L^1(X, \mu)$ as a specific integrable function, with the understanding that we identify it as an element of $L^1(X, \mu)$ with any other measurable function which agrees with it $\mu$-a.e.

- Just as with $L^1(\mathbb{R}^d, m)$, the Riesz-Fischer holds, i.e. $L^1(X, \mu)$ is a Banach space with respect to the norm

$$\|f\|_{L^1(X, \mu)} := \int |f(x)| \ d\mu(x).$$