A measure space, \((X, \mathcal{M}, \mu)\) consists of three things:

1. A set \(X\) (with no structure on it required);
2. A \(\sigma\)-algebra \(\mathcal{M}\) of subsets of \(X\);
3. A function \(\mu\) on the \(\sigma\)-algebra, which we call a measure, defined by the following properties:
   - \(\mu : \mathcal{M} \rightarrow [0, \infty]\)
   - For all pairwise disjoint sequences of sets \(E_1, E_2, \ldots\) in \(\mathcal{M}\),
     \[
     \mu \left( \bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} m(E_n).
     \]

We refer to the sets in \(\mathcal{M}\) as the \(\mu\)-measurable sets.

If all subsets of sets in \(\mathcal{M}\) having measure 0 are also in \(\mathcal{M}\), then we say that the measure space is complete, or sometimes we just say that \(\mu\) is complete.
Definition of measure space

Abstract measure space (continued)

- If \( \mu(X) < \infty \), we say that \( \mu \) is a **finite measure**. If \( \mu(X) = 1 \) we say that \( \mu \) is a **probability measure**.

- If either \( \mu \) is a finite measure, or \( \mu(X) = \infty \) but the measure space has the additional property that there exists a sequence of sets \( \{E_k\}_{k=1}^{\infty} \) in \( \mathcal{M} \) such that \( X = \bigcup_{k=1}^{\infty} E_k \) and \( m(E_k) < \infty \) for each \( k \), then we say that it is a **\( \sigma \)-finite measure space**.

- If \( X \) is a metric space (or any topological space), the **Borel \( \sigma \)-algebra** on \( X \), denoted by \( \mathcal{B}(X) \), is the smallest \( \sigma \)-algebra containing all of the open subsets of \( X \). Then if \((X, \mathcal{B}(X), \mu)\) is a measure space, we call \( \mu \) a **Borel measure on \( X \)**.

- **Note:** Typically Borel measures are not complete, but there is a fairly simple process for enlarging the Borel \( \sigma \)-algebra and extending \( \mu \) to this larger \( \sigma \)-algebra so as to obtain a complete measure space, known as the completion of \((X, \mathcal{B}(X), \mu)\).
The measure spaces which are \( \sigma \)-finite and complete are the ones for which we can expect the theory we developed for the Lebesgue measure to extend in a similar way.

Thus for such measure spaces we can expect to be able to develop a similar theory of integral (for which Fatou’s lemma, the MCT and the DCT theorem, theory of \( L^1(X, \mu) \) and more generally \( L^p(X, \mu) \) all work well), and also a theory of product measures which allows us to formulate and prove Fubini’s theorem and the Fubini-Tonelli theorem.
A few examples of measure spaces

Example 1 (Lebesgue measure):

- One example of a complete, $\sigma$-finite measure space is $(\mathbb{R}^d, \mathcal{M}, m)$ where $\mathcal{M}$ is the set of Lebesgue measurable sets and $m$ is Lebesgue measure.

- Slightly more generally, if $E$ is any Lebesgue measurable subset of $\mathbb{R}^d$, then the restriction of $m$ to $E$ is a measure space, with $X = E$ and $\mathcal{M}$ the $\sigma$-algebra of measurable subsets of $E$, i.e. $\mathcal{M} = \{E \cap F : F$ Lebesgue measurable on $\mathbb{R}^d\}$.

- This is a finite measure if and only if $m(E) < \infty$. 


Example 2 (Measures “absolutely continuous with respect to Lebesgue measure”)

- Let \( f \) be any non-negative locally integrable function on \( \mathbb{R}^d \). Then the function \( \mu \) defined on the Lebesgue measurable subsets of \( \mathbb{R}^d \), defined by

\[
\mu(E) := \int_E f(y) dm(y)
\]

is a measure on \( \mathbb{R}^d \).

- We call such a measure “absolutely continuous” because it has the property that

\[
(\forall E \in \mathcal{M})[m(E) = 0 \implies \mu(E) = 0].
\]

We will return to study this idea of absolutely continuous measures in detail in a later section.

- Since \( f \) is locally integrable, \( \mu \) is \( \sigma \)-finite. The measure \( \mu \) is a finite measure if and only if \( f \) is integrable.
Example 3 (Counting measure)

- Let $X = \{x_i\}_{i \in I}$ be any countable set and $\{m_i\}_{i \in I}$ any countable collection of non-negative numbers, where the indexing set $I$ is either the positive integers or a non-empty finite set.
- We take $\mathcal{M}$ to be the power set of $X$, i.e. the set of all subsets of $X$.
- For each element $E$ of $\mathcal{M}$ we define $\mu(E) = \sum_{\{i : x_i \in E\}} m_i$.
- So we are viewing each $m_i$ as representing the measure, or the “mass” of $x_i$.
- Then $(X, \mathcal{M}, \mu)$ is a $\sigma$-finite measure space for which $\mu$ is finite if and only if $\{m_i\}_i$ is a summable sequence.
- In the special case that every $m_i$ is 1, then we call $\mu$ the counting measure on $X$, since in that case the measure of a given set is just the number of elements in that set.
Example 4 (The Dirac measure)

- Let \( x_0 \in \mathbb{R}^d \). The \textbf{Dirac measure} with support at \( x_0 \), denoted by \( \delta_{x_0} \), is the measure on the collection of all subsets of \( \mathbb{R}^d \), defined by

\[
\delta_{x_0}(E) = \begin{cases} 
1 & \text{if } x_0 \in E \\
0 & \text{if } x_0 \notin E
\end{cases}
\]

- Note that the measure “lives” on the point set \( \{x_0\} \) (since the measure of the complement is 0).
- Thus it is an example of a measure which is said to be \textit{“singular with respect to Lebesgue measure”} because it has its support on a set of Lebesgue measure 0.
- We will study \textit{“mutually singular measures”} in greater detail in a later section.
Example 5 (Lebesgue-Stieltjes measures on $\mathbb{R}$)

- Let $\mathcal{B} = \mathcal{B}(\mathbb{R})$ denote the $\sigma$-algebra of Borel subsets of $\mathbb{R}$.
- Let $F : \mathbb{R} \to \mathbb{R}$ be any function which is increasing and right continuous, i.e.
  1. $(\forall x, y \in \mathbb{R})[x \leq y \implies F(x) \leq F(y)];$
  2. $(\forall x \in \mathbb{R})[F(x) = \lim_{y \to x^+} F(y)]$

- Define $\mu$ on intervals of the form $(a, b]$ by
  $$\mu((a, b]) = F(b) - F(a).$$

It is shown in a later section that $\mu$ can be extended in a unique way to a Borel measure on $\mathbb{R}$. And conversely, every Borel measure on $\mathbb{R}$ which is finite on bounded intervals can be obtained in this way for a suitable function $F$.

- We call any such measure a \textit{Lebesgue-Stieltjes measure on $\mathbb{R}$}.
- For example, the Lebesgue measure on $\mathbb{R}$ is a Lebesgue-Stieltjes measure corresponding to the function $F(x) = x$. and the Dirac delta measure $\delta_{x_0}$ is a Lebesgue-Stieltjes measure with
  $$F(x) = \begin{cases} 
  0 & \text{if } x < x_0 \\
  1 & \text{if } x \geq x_0 
  \end{cases}$$
In developing the Lebesgue measure on $\mathbb{R}^d$ we used the following outline:

1. We decided we wanted the theory to be determined by cubes, starting with the idea that the measure of a cube should turn out to be its volume;

2. We defined a function which we called exterior measure, which for each set $E$ produced the number $m^*(E)$, determined by coverings of $E$ by countably many cubes, and the volume of each of those cubes;

3. We proved that the exterior measure of a cube is its volume;

4. We defined what it meant for a set $E$ to be Lebesgue measurable, where the definition involved the exterior measure $m^*$ and the underlying topology of $\mathbb{R}^d$.

5. We proved that the set of Lebesgue measurable sets forms a $\sigma$-algebra containing all the open sets, and that the restriction of $m^*$ to those measurable sets is a measure.
For the general theory, we articulate two main ideas, both inspired in some way by the Lebesgue theory, but used quite differently.

One of the main differences is to modify and isolate the role played by analogues of the cubes.

The two ideas concern

1. **Exterior measure:** Develop a general theory of exterior measure, completely independent of any special kind of set (such as the cubes in the theory of Lebesgue measure).

2. **Extension of a premeasure:** By making use of the abstract theory of exterior measure, show that if we have a special kind of set function (called a premeasure) defined on a family of sets smaller than a $\sigma$-algebra, then it can be extended to a measure on a $\sigma$-algebra of sets. This is the step which relates to the idea of prescribing the measure on a special family of sets, such as cubes.
Exterior measure:

- We develop an abstract theory of exterior measure on $X$ (without imposing any special structure on $X$), and use it to define what we mean by a measurable set.
- The definition of exterior measure is independent of any special kind of set (such as a cube).
- The idea of measurable set will be independent of any topology, and is instead inspired by the desired additivity property of measure.
- One can then show that the restriction of the exterior measure to the set of measurable sets makes it a measure.
- One interesting example of an exterior measure comes from imposing a metric space structure on $X$. And if the exterior measure happens to have an additional property, which makes it into an “exterior metric measure”, then it can be shown that the collection of measurable sets necessarily contains all of the Borel sets, and furthermore we can equate the abstract theory of measurable set to the familiar one involving open sets.
Extension of premeasures:

- In order to take advantage of the abstract theory of exterior measure described above, we can start with a set function $\mu_0$, known as a premeasure, defined on an algebra of sets (rather than on a $\sigma$-algebra of sets).

- The members of the algebra (roughly) play the role of the cubes in the theory of the Lebesgue measure.

- Then the idea is to figure out how to extend the premeasure to a measure on the smallest $\sigma$-algebra containing the above algebra.

- This is done by associating to $\mu_0$ an exterior measure, and then making use of the general theory of exterior measure.

- The method works because of the very careful choice of axioms used to define what is a premeasure.
Exterior measures

Definition of exterior measure

Let $X$ be any set. An exterior measure on $X$ is a function $\mu_* : \mathcal{P}(X) \to [0, \infty]$ satisfying the following three properties:

1. $\mu_*(\emptyset) = 0$
2. (monotonicity) $(\forall E_1, E_2 \in \mathcal{P}(X))[E_1 \subset E_2 \implies \mu_*(E_1) \leq \mu_*(E_2)]$
3. (countable subadditivity) For any countable collection $E_1, E_2, \ldots$ of subsets of $X$,
   \[
   \mu_* \left( \bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} \mu_*(E_k).\]

Of course the exterior measure we studied with Lebesgue measure is an example of an exterior measure.

Recall we defined a set $E$ to be Lebesgue measurable if for any $\varepsilon > 0$ there exists an open set $O$ such that $E \subset O$ and $\mu_*(O \setminus E) < \varepsilon$.

In the abstract setting $X$ is not assumed to have any topology, so in order to define measurability, we need to take a different approach.
The approach we take is due to Carathéodory, and is motivated by the idea of additivity.

**Definition of $\mu_\ast$-measurability**

A subset $E$ of $X$ is called **Carathéodory measurable**, or just **$\mu_\ast$-measurable** if for every subset $A$ of $X$ it is true that

$$
\mu_\ast(A) = \mu_\ast(E \cap A) + \mu_\ast(E^c \cap A).
$$

- We might express the above property as "$E$ nicely separates every subset of $X$", and refer to the property as "the nice separation property of $E$".
- Note that in order to confirm a set $E$ is $\mu_\ast$-measurable it is only necessary to prove that for every subset $A$ of $X$ we have

$$
\mu_\ast(A) \geq \mu_\ast(E \cap A) + \mu_\ast(E^c \cap A),
$$

since the reverse inequality always holds by countable subadditivity.
Main result on exterior measures

Theorem

Given an exterior measure $\mu_*$ on a set $X$, the set of $\mu_*$-measurable sets forms a $\sigma$-algebra and the restriction of $\mu_*$ to the $\mu_*$-measurable sets is a measure. Furthermore $\mu_*$ is complete.

Note: This is basically Theorem 1.1 in the text, except we add to that theorem the assertion that $\mu_*$ is complete.

Proof.
Connection with our old definition of measurability

In order to relate this new idea of $\mu_*$-measurable to our original definition involving open sets, we shall impose a metric space structure on $X$.

With this metric space structure, it makes sense to talk about open sets, and therefore about Borel sets.

In order to get interesting results, we shall impose the following condition which relates the metric and the exterior measure.

---

**Metric exterior measure**

Let $\mu_*$ be an exterior measure on a set $X$, and assume in addition $X$ is a metric space with metric $d$.

We say that $\mu_*$ is a “metric exterior measure” provided it is “additive on well-separated sets”, that is for all subsets $A, B$ of $X$,

$$\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$$

provided $d(A, B) > 0$

where

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$
Connection with our old definition of measurability

Theorem 1.2

Let $X$ be a metric space and $\mu_*$ a metric exterior measure on $X$, i.e. an exterior measure on $X$ which is additive on well-separated sets. Then every Borel set is $\mu_*$-measurable. In particular $\mu_*$ can be viewed as a Borel measure on $X$.

Proof.
Connection with our old definition of measurability

Proposition 1.3

Let $\mu$ be any Borel measure on a metric space $X$. Suppose in addition that $\mu(K) < \infty$ for every compact subset $K$ of $X$. Then for any Borel set $E$ and any $\varepsilon > 0$ there exists an open set $O$ and closed set $F$ such that

$$
\mu(O \setminus E) < \varepsilon \text{ and } \mu(E \setminus F) < \varepsilon.
$$

Proof.
Return to the general framework: The Carathéodory Extension Theorem

**Algebras and premeasures**

- Let $X$ be any set. An **algebra on $X$** is a nonempty collection of sets $\mathcal{A}$ which is closed under the operation of finite unions, finite intersections, and complements.

- A **premeasure on $\mathcal{A}$** is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ satisfying the following two properties:
  - $\mu_0(\emptyset) = 0$;
  - (**$\sigma$-additivity**) Let $E_1, E_2, \ldots$ be any countable collection of pairwise disjoint sets in $\mathcal{A}$. If it happens to be true that $\bigcup_k E_k$ is in the algebra, then
    \[
    \mu_0 \left( \bigcup_k E_k \right) = \sum_k \mu_0(E_k).
    \]

The definition of premeasure is not what you perhaps expected.

But what we want to be able to prove is that it is always possible to extend a premeasure to a measure on the $\sigma$-algebra generated by the algebra, i.e. the smallest $\sigma$-algebra which contains the algebra. The magic ingredient in proving that this is possible is precisely the $\sigma$-additivity property, so the definition is really quite subtle.
Lemma 1.4

If $\mu_0$ is a premeasure on an algebra $\mathcal{A}$, define $\mu_*$ on any subset $E$ of $X$ by

$$
\mu_*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(E_j) : E \subset \bigcup_{j=1}^{\infty} E_j, \text{ where } E_j \in \mathcal{A} \text{ for all } j \right\}.
$$

Then $\mu_*$ is an exterior measure on $X$ with the following two additional properties:

- $\mu_*(E) = \mu_0(E)$ for all $E \in \mathcal{A}$
- All sets in $\mathcal{A}$ are $\mu_*$-measurable.

Proof.
Return to the general framework: The Carathéodory Extension Theorem

Theorem 1.5

Suppose that $\mathcal{A}$ is an algebra of sets in $X$, $\mu_0$ a premeasure on $\mathcal{A}$, and $\mathcal{M}$ the smallest $\sigma$-algebra that contains $\mathcal{A}$. Then there exists a measure $\mu$ on $\mathcal{M}$ which extends $\mu_0$. Furthermore this measure is unique on sets of finite measure. Thus the extension is unique if $\mu$ is $\sigma$-finite.

Proof.
One simple, but significant, example of an algebra of sets on \( \mathbb{R} \) is the set of finite unions of left open and right closed intervals, together with open intervals unbounded on the right.

Basic set theory allows us to prove that the \( \sigma \)-algebra generated by this algebra is the same as the one generated by all open intervals, and that is the same as all of the Borel sets.

By defining \( \mu_0((a, b]) = b - a \), we might expect to be able to prove that \( \mu_0 \) is a premeasure, and so by Theorem 1.5 extends uniquely to a Borel measure on \( \mathbb{R} \). Of course the extension is the Lebesgue measure.

A similar method can be used to construct Lebesgue-Stieltjes measures on \( \mathbb{R} \) by instead defining

\[
\mu_0((a, b]) = F(b) - F(a)
\]

for a given increasing and right continuous function \( F \) on \( \mathbb{R} \).

These ideas are developed in detail in section 3 of Chapter 6.