Finite Element Methods

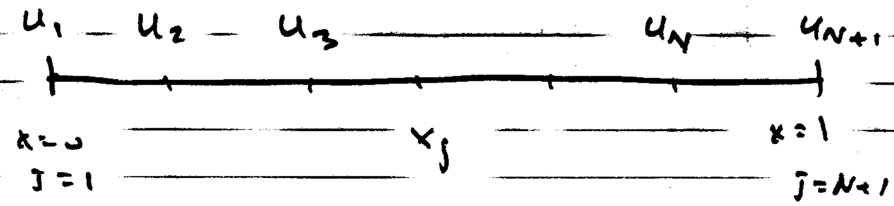
- We'll focus here on 1D problems but note that FEM are very powerful methods in higher dimensions.

We'll start our discussion in the context of the following linear two-point boundary value problem

$$
\begin{cases}
-\dfrac{d}{dx}\left[a(x)\dfrac{du}{dx}\right] + b(x)u(x) = f(x) & 0 \le x \le 1 \\
u(0) = \alpha \\
u(1) = \beta
\end{cases}
$$

We'll consider various approaches to solving (approximately) this BVP that will lead us to the finite element method. [Collocation, Method of Weighted Residuals, Galerkin, ...]

Unlike the finite difference approach where we attempt to find approximations to the solution at a discrete set of points, e.g.



so $\quad u_j \approx u(x_j) \quad j=1,2,\dots,N+1$

The idea here is to approximate the solution to the BVP by a linear combination of <u>functions</u> in a finite dimensional space spanned by basis functions $\phi_\ell(x)$

$$u(x) \approx \sum_{\ell=0}^{m} c_\ell \phi_\ell(x)$$

<u>For example</u>, the finite dimensional space could be the space of polynomials of degree 3 or less so that we could use

$$\phi_\ell(x) = x^{\ell-1}$$

$$\ldots \quad \{1, x, x^2, x^3\}$$

so that

$$u(x) \approx c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

Here, we would be attempting to approximate the solution to the BVP by a cubic polynomial — we'd need some method to determine the coefficients c_ℓ.

For the Finite Element Method discussion we'll eventually modify this notation slightly by writing

$$u(x) \approx \phi_0(x) + \sum_{\ell=1}^{m} c_\ell \phi_\ell(x) \equiv u_m(x)$$

e.g. a linear function

where $\phi_0(x)$ is a function that satisfies b.c. $\phi(0)=\alpha$ and $\phi_0(1)=\beta$ and the functions ϕ_ℓ satisfy $\phi_\ell(0)=\phi_\ell(1)=0$.

Here c_ℓ $\ell = 1, 2, \ldots, m$ are real constants to be determined. Further ~~$\{\phi_1, \phi_2, \ldots, \phi_m\}$~~ the functions $\phi_1, \phi_2, \ldots, \phi_m$ are linearly independent functions that span a finite dimensional space $\mathring{H}_m$

$$\mathring{H}_m \equiv \text{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$$ (see Iserles, p. 172)

<u>Note</u>

- $u_m - \phi_0 \in \mathring{H}_m$
- every element of $\mathring{H}_m$ satisfies zero b.c. at $x=0$ and $x=1$.

So, the first key principle for FEM is

$$\boxed{\text{Approximate the solution in a finite-dimensional space}}$$

The next issue is to think about choosing the coefficients c_ℓ in a suitable way.

- this questions leads us to collocation method, MWR and Galerkin methods first...

Collocation Method

- this method is related to the ideas of interpolation of a known function but instead of requiring that the interpolant matches the function value at collocation points we require that the ODE be satisfied at the collocation points (the collocation points can be equally-spaced or not).

EXAMPLE

$$\begin{cases} u'' + u = x^4 \\ u(0) = 0 \\ u(1) = 1 \end{cases}$$

Comment: this has exact sol.

$$u(x) = -24\cos x + \frac{24\cos(1) - 12}{\sin(1)} \sin x + x^4 - 12x^2 + 24$$

Seek a solution of the form

$$u(x) \approx \sum_{\ell=1}^{N} c_\ell \, \phi_\ell(x)$$

where ϕ_ℓ are basis functions in some finite dimensional space (dimension N) to be specified and c_ℓ are constants to be determined.

Comment

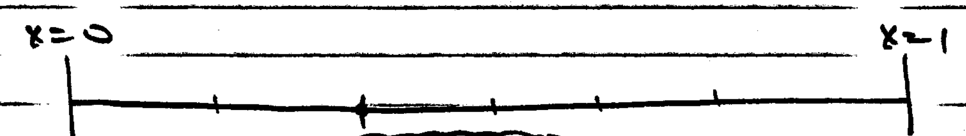
- in this example we are not imposing any a priori form on the functions $\phi_\ell(x)$ to match boundary condition (for ease of calculation in this example).

So in principle we would like to have

$$\left\{\begin{aligned}
&\sum_{\ell=1}^{N} c_\ell \phi_\ell''(x) + \sum_{\ell=1}^{N} c_\ell \phi_\ell(x) = x^4 \qquad \circledast\,! \\
&\sum_{\ell=1}^{N} c_\ell \phi_\ell(0) = 0 \\
&\sum_{\ell=1}^{N} c_\ell \phi_\ell(1) = 1
\end{aligned}\right.$$

- Note that unless we are very lucky with our choice of functions $\phi_\ell(x)$ in general we cannot acheive the condition $\circledast$ at all values of x (i.e. we won't be able to satisfy the ODE everywhere). The idea of collocation is to "enforce" the ODE at collocation ~~pts~~ points to get a closed system for the coefficients c_ℓ. The b.c. determine two such conditions and so we need $N-2$ other conditions.

Introduce collocation pts.



$$x_j \equiv \frac{(j-1)}{N-1}$$

and "enforce" the ODE at the $N-2$ interior pts.

That is

$$\begin{cases} \displaystyle\sum_{\ell=1}^{N} c_\ell \phi_\ell''(x_j) + \sum_{\ell=1}^{N} c_\ell \phi_\ell(x_j) = x_j^4 & j=2,3,\ldots,N-1 \\ \displaystyle\sum_{\ell=1}^{N} c_\ell \phi_\ell(0) = 0 \\ \displaystyle\sum_{\ell=1}^{N} c_\ell \phi_\ell(1) = 1 \end{cases}$$

this is a system of N equations for N unknowns (c_ℓ) once we specify the basis functions

Let's consider the <u>choice of basis</u> functions to be ...

monomial basis (other choices are possible... more later)

$$\phi_\ell(x) = x^{\ell-1} \qquad (\text{i.e. } 1, x, x^2, \ldots, x^{N-1}) \quad \ell=1,2,\ldots,N$$

then

$$\phi_\ell'(x) = (\ell-1)\,x^{\ell-2}$$

$$\phi_\ell''(x) = (\ell-2)(\ell-1)\,x^{\ell-3}$$

So...

$$j=1: \quad \sum_{l=1}^{N} c_l \, 0^{l-1} = 0 \quad \Rightarrow \quad \boxed{c_1 = 0}$$

$$j=2,\dots,N-1: \quad \boxed{\sum_{l=1}^{N} c_l (l-2)(l-1)\, x_j^{l-3} + \sum_{j=1}^{N} c_l\, x_j^{l-1} = x_j^4}$$

$$j=N: \quad \boxed{\sum_{l=1}^{N} c_l \cdot 1^{l-1} = 1}$$

This is a liner system of the form

$$\begin{array}{l} j=1 \\ j=2 \\ \vdots \\ \\ \\ \\ \\ j=N-1 \\ j=N \end{array}
\begin{bmatrix} 1 \quad 0 \quad 0 \quad \cdots \quad 0 \\ \\ \text{metrix with entries } A_{jl} \\ \\ A_{jl} = (l-2)(l-1)\, x_j^{l-3} + x_j^{l-1} \\ l = 1,2,\dots,N \\ j = 2,\dots,N-1 \\ \\ 1 \quad 1 \quad 1 \quad \cdots \quad 1 \end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ \\ \\ \vdots \\ c_{N-1} \\ c_N \end{bmatrix}
=
\begin{bmatrix} 0 \\ x_2^4 \\ x_3^4 \\ \vdots \\ \\ \\ \vdots \\ x_{N-1}^4 \\ 1 \end{bmatrix}$$

Suppose $N=3$

$$x_j = \frac{j-1}{3-1} = \frac{j-1}{2} \qquad \boxed{x_1 = 0,\ x_2 = \tfrac{1}{2},\ x_3 = 1}$$

$$\begin{array}{c} \\ j=1 \\ j=2 \\ j=3 \end{array} \begin{array}{c} \begin{array}{ccc} \ell=1 & \ell=2 & \ell=3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & 2+\left(\frac{1}{2}\right)^2 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ (1/2)^4 \\ 1 \end{bmatrix}$$

$\hookrightarrow$ solving gives $\boxed{c_1 = 0}$ $\boxed{c_2 = 5/4}$ $\boxed{c_3 = -1/4}$

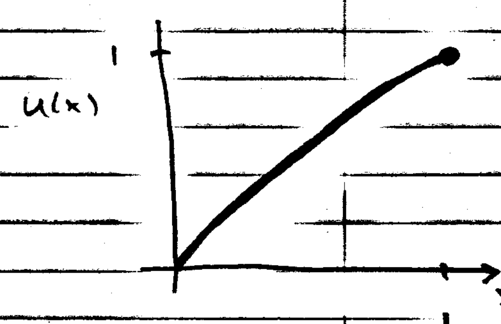
So

$$u(x) \approx \sum_{\ell=1}^{3} c_\ell x^{\ell-1} = 0 + \frac{5}{4}x - \frac{1}{4}x^2 \equiv v(x)$$

If we define the error as $\dfrac{\| v(x) - u_e(x) \|_2}{\| u_e(x) \|_2}$ $\quad$ $u_e(x)$ = exact sol.

we find

N	error
3	1.5×10^{-2}
6	1.6×10^{-4}
12	6.6×10^{-13}



see bvp_coll.m

the solution in this example is not far from linear so perhaps the good results are not surprising

EXAMPLE (Collocation - nonlinear)

$$\begin{cases} u'' + u^2 = x \\ u(0) = 0 \\ u(1) = 1 \end{cases}$$

Let $u(x) \approx \sum_{\ell=1}^{N} c_\ell \phi_\ell(x)$

using the same collocation pts. as in the previous example...

We require

$j=1$ $\quad F_1(\vec{c}) = \sum_{\ell=1}^{N} c_\ell \phi_\ell(0) = 0$

$j=2,\dots,N-1$ $\quad F_j(\vec{c}) = \sum_{\ell=1}^{N} c_\ell \phi_\ell''(x_j) + \left(\sum_{\ell=1}^{N} c_\ell \phi_\ell(x_j)\right)^2 - x_j = 0$

$j=N$ $\quad F_N(\vec{c}) = \sum_{\ell=1}^{N} c_\ell \phi_\ell(1) - 1 = 0$

where $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$ is the unknown vector and

$\vec{F}(\vec{c}) = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{bmatrix}$ is the function (nonlinear)

We need to find $\vec{c}$ such that $\vec{F}(\vec{c}) = \vec{0}$.

(e.g. apply Newton's method)

So iterate the linear solve for $\vec{w}^{K+1}$

$$J_F(\vec{w}^K)\left(\vec{w}^{K+1} - \vec{w}^K\right) = -\vec{F}(\vec{w}^K)$$

given $\vec{w}^K$.

Note that the Jacobian J_F is defined by

(row 1) $j=1$: $$\frac{\partial F_1}{\partial c_\ell} = \phi_\ell(0)$$

row (2 … N-1) $j = 2 \dots N-1$ $$\frac{\partial F_j}{\partial c_\ell} = \phi''_\ell(x_j) + 2\underbrace{\left(\sum_{i=1}^{N} c_i \phi_i(x_j)\right)}_{\text{separate summation}} \phi_\ell(x_j)$$

row N $$\frac{\partial F_N}{\partial c_\ell} = \phi_\ell(1)$$

See bvp_coll_nl.m (main w/ Newton solver)
bvp_JF_coll.m (function + Jacobian)