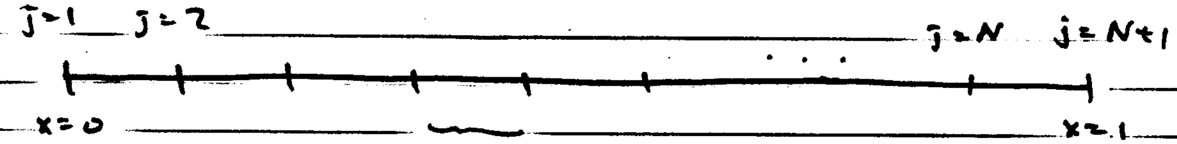
Finite Differences -

A nonlinear example in 1D:

Consider

$$\begin{cases} u'' = -e^{u+1} & \text{on } 0 < x < 1 \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

Discretize



$\Delta x = \frac{1}{N}$ $\qquad x_j = \frac{j-1}{N}$

Approximate $u''(x_j) \approx \dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$

So we wish to solve for unknowns $u_1, u_2, \ldots, u_N, u_{N+1}$

using

$$\begin{cases} u_0 = 0 \\ u_{j+1} - 2u_j + u_{j-1} = -h^2 e^{u_j + 1} & j = 2, 3, \ldots, N \\ u_{N+1} = 0 \end{cases}$$

This is not a linear system — its a big nonlinear system of equations...

Basically, we have $\vec{F}(\vec{u}) = \vec{0}$ where

$$\vec{u} = \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_N \end{bmatrix} \quad \text{(interior pts.)} \qquad u_1 = 0, \; u_{N+1} = 0$$

and

$$j=2 \qquad F_2(\vec{u}) = u_3 - 2u_2 + \cancel{u_1}^{=0 \text{ (b.c.)}} + h^2 e^{u_2+1}$$

$$j=3 \qquad F_3(\vec{u}) = u_4 - 2u_3 + u_2 + h^2 e^{u_3+1}$$

$$\vdots$$

$$j=N \qquad F_N(\vec{u}) = \cancel{u_{N+1}}^{0 \text{ (b.c.)}} - 2u_N + u_{N-1} + h^2 e^{u_N+1}$$

N−1 equations for N−1 unknowns

We could solve this problem using Newton's method ...

$$\vec{w}^{(k+1)} = \vec{w}^{(k)} - J_f(\vec{w}^{(k)})^{-1} \cdot \vec{f}(\vec{w}^{(k)}) \qquad k = 1, 2, 3, \ldots$$

where $\vec{w}^{(k)} \rightarrow \vec{u}$ (if successful!)

$$J_f = \begin{bmatrix} \frac{\partial F_2}{\partial u_2} & \frac{\partial F_2}{\partial u_3} & \cdots & \frac{\partial F_2}{\partial u_N} \\ \frac{\partial F_3}{\partial u_2} & \frac{\partial F_3}{\partial u_3} & \cdots & \vdots \\ \vdots & & & \vdots \\ \frac{\partial F_N}{\partial u_2} & \frac{\partial F_N}{\partial u_3} & \cdots & \frac{\partial F_N}{\partial u_N} \end{bmatrix}.$$

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$$J_f(\vec{w}^{(k)}) = \begin{bmatrix} \left(-2 + h^2 e^{w_2^{(k)}+1}\right) & 1 & 0 & \cdots & 0 \\ 1 & \left(-2 + h^2 e^{w_3^{(k)}+1}\right) & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & \left(-2 + h^2 e^{w_N^{(k)}+1}\right) \end{bmatrix}$$

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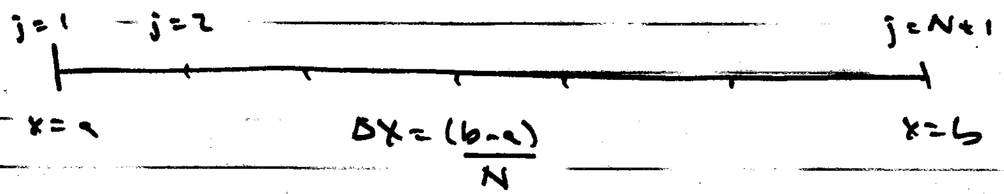
Finite Differences (Neumann b.c.)

$(u'' + \epsilon u' + u = 0)$

Consider

$$\begin{cases} u'' = f(x,u,u') = -\epsilon u' - u & \text{(linear example)} \\ u(a) = \alpha \\ u'(b) = \beta \longleftarrow \text{Neumann Condition at } x=b \end{cases}$$

Discretize



$$x_j = a + \frac{j-1}{N}(b-a)$$

Approximate — Second order Accurate

$$u_j'' = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} \qquad (+O(\Delta x^2))$$

$$u_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x} \qquad (+O(\Delta x^2))$$

So the ODE looks like

$$\boxed{\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} + \epsilon\frac{u_{j+1} - u_{j-1}}{2\Delta x} + u_j = 0}$$

what values of j apply here?

Let's first think about B.C.

- We have $u(a)=\alpha$ so $\boxed{u_1=\alpha}$

- We want to use some type of approximation for $u'(b)$ that has the same accuracy as the approximations we've used in the ODE.

$\boxed{\text{Option 1}}$ - one sided

- use one sided, second-order accurate difference formula for $u'(b)$

$u_{N+1} = u(x)$

$u_N = u(x-\Delta x) = u(x) - \Delta x\, u'(x) + \frac{1}{2}\Delta x^2 u''(x) - \frac{1}{3!}\Delta x^3 u'''(x) + \dots$

$u_{N-1} = u(x-2\Delta x) = u(x) - 2\Delta x\, u'(x) + 2\Delta x^2 u''(x) - \frac{8}{3!}\Delta x^3 u'''(x) + \dots$

$4u_N - u_{N-1} = 3u_{N+1} - 2\Delta x\, u'(x) + O(\Delta x^3)$

so

$$\boxed{u'_{N+1} \approx \frac{3u_{N+1} - 4u_N + u_{N-1}}{2\Delta x} + O(\Delta x^2)}$$

Here: our equations for unknowns are:

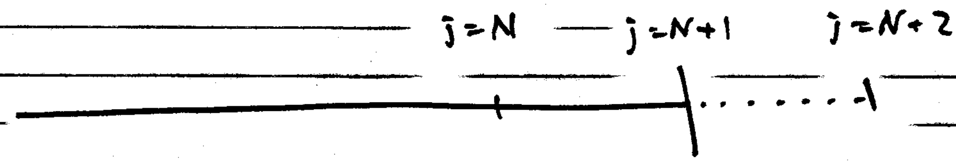
$u_1 = \alpha$ $(j=1)$

$\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} + \epsilon \frac{u_{j+1} - u_{j-1}}{2\Delta x} + u_j = 0$ interior pts $j=2,3,\dots,N$

$\frac{3u_{N+1} - 4u_N + u_{N-1}}{2\Delta x} = \beta$ $(j=N+1)$

$\boxed{\text{option 2}}$ – ghost point

Introduce a 'ghost' point at $j = N+2$



The b.c. $u'(b) = \beta$ approximated using a <u>central</u> difference formula then gives

$$\frac{u_N - u_{N+2}}{2\Delta x} = \beta$$

That is, the value of the ghost point can be written in terms of one of the other unknowns (u_N) and so to enforce the b.c. we can use

$$\boxed{u_{N+2} = u_N - 2\Delta x \cdot \beta}$$

Then, instead of applying the ODE at the interior pts. only we apply it at $j = 2, 3, \ldots, N, \underline{\underline{N+1}}$ noting that we actually know u_{N+2}

$$\left\{ \begin{array}{l} \dfrac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} + \epsilon \dfrac{u_{j+1} - u_{j-1}}{2\Delta x} + u_j = 0 \qquad j = 2, 3, \ldots, N+1 \\ \text{along with} \\ \quad u_1 = \alpha \quad \text{and} \quad u_{N+2} = u_N - 2\Delta x \cdot \beta \end{array} \right.$$

Our "solution" to the BVP is still

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_N \\ u_{N+1} \end{bmatrix}$$

... we've just used the ghost point u_{N+2} as a means to finding the unknowns we wanted.

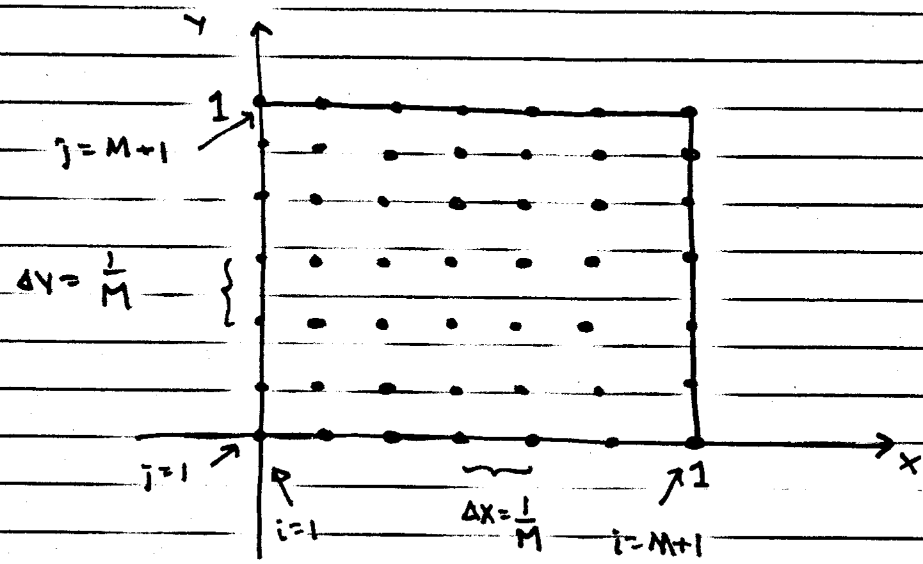
Consider the Poisson Eq. in 2D

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \qquad \text{in } \Omega$$

subject to Dirichlet conditions on the boundary $\partial\Omega$

$$u(x,y) = \phi(x,y) \quad \text{on } \partial\Omega$$

For starters, lets consider Ω to be the unit square



Using a second-order finite difference approximation for the second derivative terms ...

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + O(\Delta x^2)$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} + O(\Delta y^2)$$

Note that these can be applied at internal points but not at the boundary (but comment on 'ghost' points)

So for $i = 2,3,\dots,M$ and $j = 2,\dots,N$ the PDE is represented by

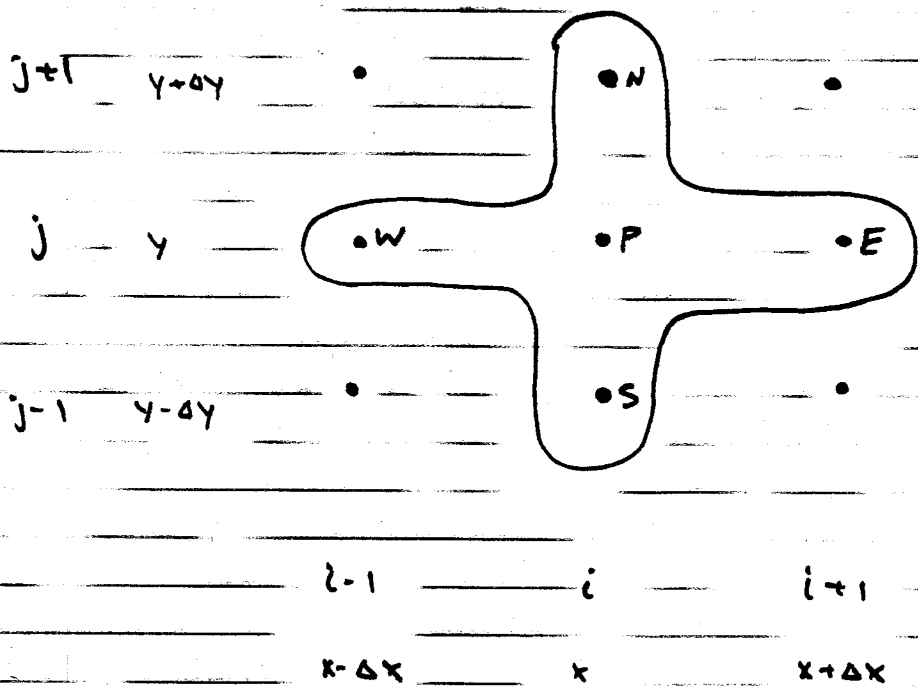
$$\boxed{\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} = f(x_i, y_j)}$$

applies at $(M-1) \times (N-1)$ interior points.

where $x_i = \dfrac{(i-1)}{M}$ $\quad i = 1,2,\dots,M+1$

$y_j = \dfrac{(j-1)}{N}$ $\quad j = 1,2,\dots,N+1$

We see that the PDE approximated at point i,j has associated with it a five point stencil



We also impose boundary conditions

$u(x,y) = \phi(x,y)$ at the boundary points

There are $\underbrace{\underbrace{2(M-1)}_{\substack{\text{bottom}\\ \text{+top}}} + \underbrace{2(N-1)}_{\text{sides}}}_{\text{excluding corners}} + \underbrace{4}_{\substack{4\\ \text{corners}}}$ boundary points

$= 2M + 2N$

This gives us a total of $\underbrace{(M-1)\times(N-1) + 2M + 2N}$ equations

$= M\cdot N - N - M + 1 + 2M + 2N$

$= M\cdot N + N + M + 1$

$= \boxed{(M+1)(N+1)}$

total # of equations (interior + boundary)

Total # of unknowns $= (M+1)\times(N+1)$

Since $\begin{cases} \nabla^2 u = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$ is a linear problem this

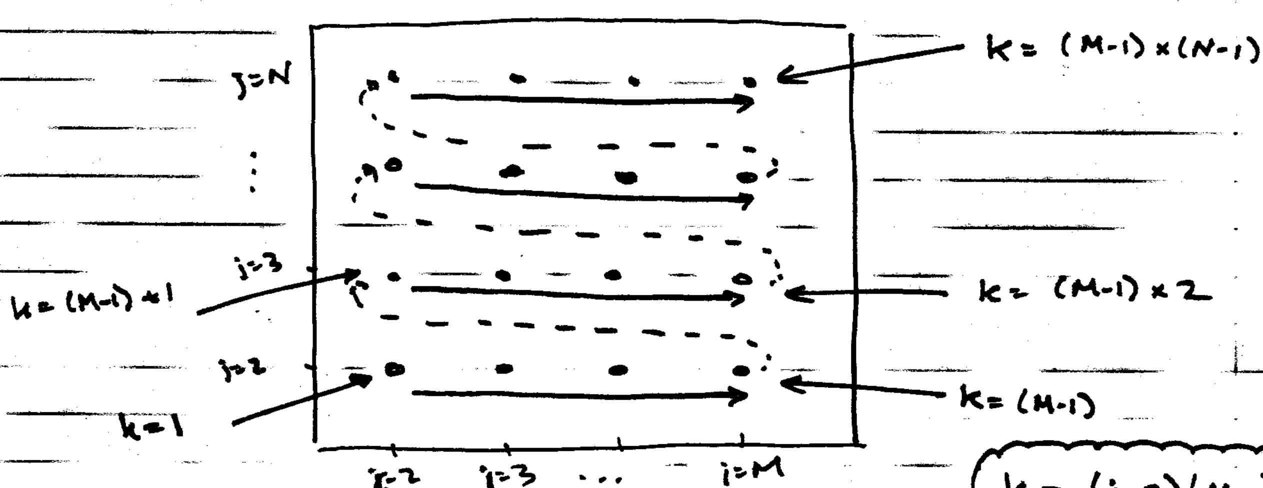
discretization gives us effectively a big linear system ($Ax=b$) to solve.

Let's sort out what the matrix/vector system is...

First, lets identify the unknowns: $u_{i,j}$ — $i = 1, 2, \dots, M+1$, $j = 1, 2, \dots, N+1$

but of these 'unknowns' we only really need to solve for the interior points $i = 2, \dots, M$
$j = 2, \dots, N$

Basically, our plan will be to list these in a vector of length $(M-1) \times (N-1)$. We'll order these from the lower left corner to the upper right corner...



$$\boxed{k \equiv (j-2)(M-1) + (i-1)}$$

$$\boxed{\begin{aligned} j &= 2, \dots, N \\ i &= 2, \dots, M \end{aligned}}$$

So define $\vec{u}$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \\ u_k \\ \vdots \\ \\ u_{(M-1)(N-1)} \end{bmatrix} = \begin{bmatrix} u_{2,2} \\ u_{3,2} \\ \vdots \\ u_{M,2} \\ \hline u_{2,3} \\ u_{3,3} \\ \vdots \\ u_{M,3} \\ \hline \vdots \\ u_{M,N} \end{bmatrix}$$

bottom interior row (M-1 unknowns)

fix

next row up ... (M-1 unknowns)

Before we write down the matrix, recall that in this particular case (Dirichlet b.c.) we know ~~the solution at all~~ the boundary points. So

$$\left.\begin{array}{ll} u_{i,1} & i=1,2,\dots,M+1 \\ u_{i,N+1} & i=1,2,\dots,M+1 \\ u_{1,j} & j=1,2,\dots,N+1 \\ u_{M+1,j} & j=1,2,\dots,N+1 \end{array}\right\}$$

all known by b.c.

~~(note corners overlap in this list)~~

Also, lets rewrite the equations for the interior pts...

$$u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + \left(\frac{\Delta x}{\Delta y}\right)^2 \left[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right] = (\Delta x)^2 f_{i,j}$$

$$u_{i+1,j} - 2\left(1 + \left(\frac{\Delta x}{\Delta y}\right)^2\right) u_{i,j} + u_{i-1,j} + \left(\frac{\Delta x}{\Delta y}\right)^2 \left[u_{i,j+1} + u_{i,j-1}\right] = (\Delta x)^2 f_{i,j}$$

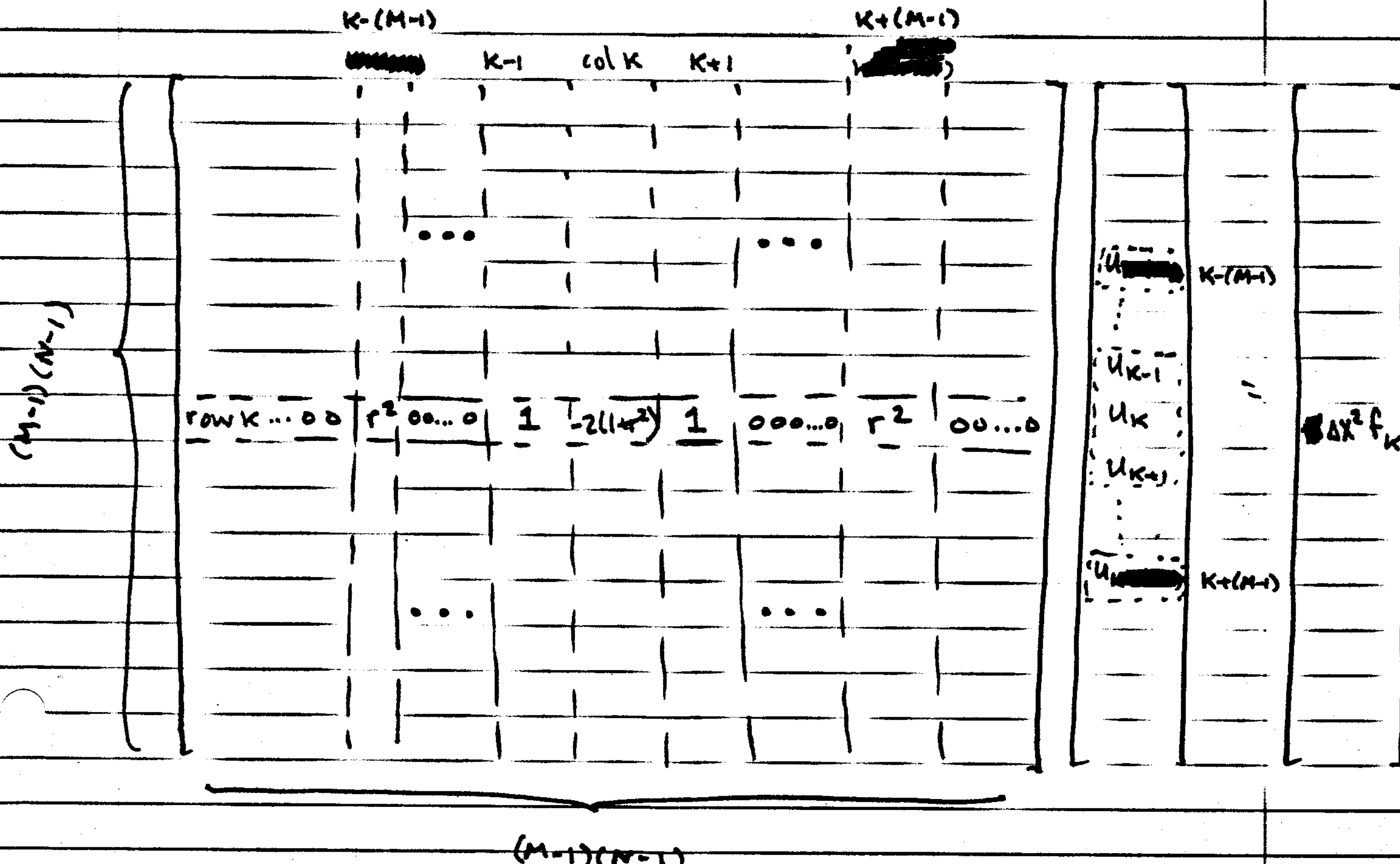
let $r \equiv \frac{\Delta x}{\Delta y}$

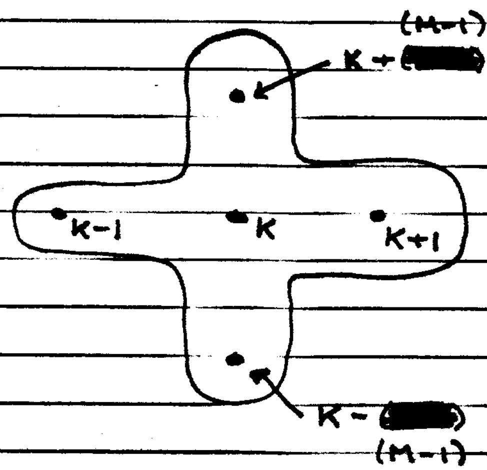
$$\boxed{u_{i+1,j} + u_{i-1,j} + r^2 u_{i,j+1} + r^2 u_{i,j-1} - 2(1+r^2)u_{i,j} = \Delta x^2 f_{i,j}}$$

for $i = 2,\dots,M$
$j = 2,\dots,N$

$$u_{K+1} + u_{K-1} + r^2 u_{K+(M+1)} + r^2 u_{K-(M+1)} - 2(1+r^2)u_K = 2\Delta x^2 f_K$$

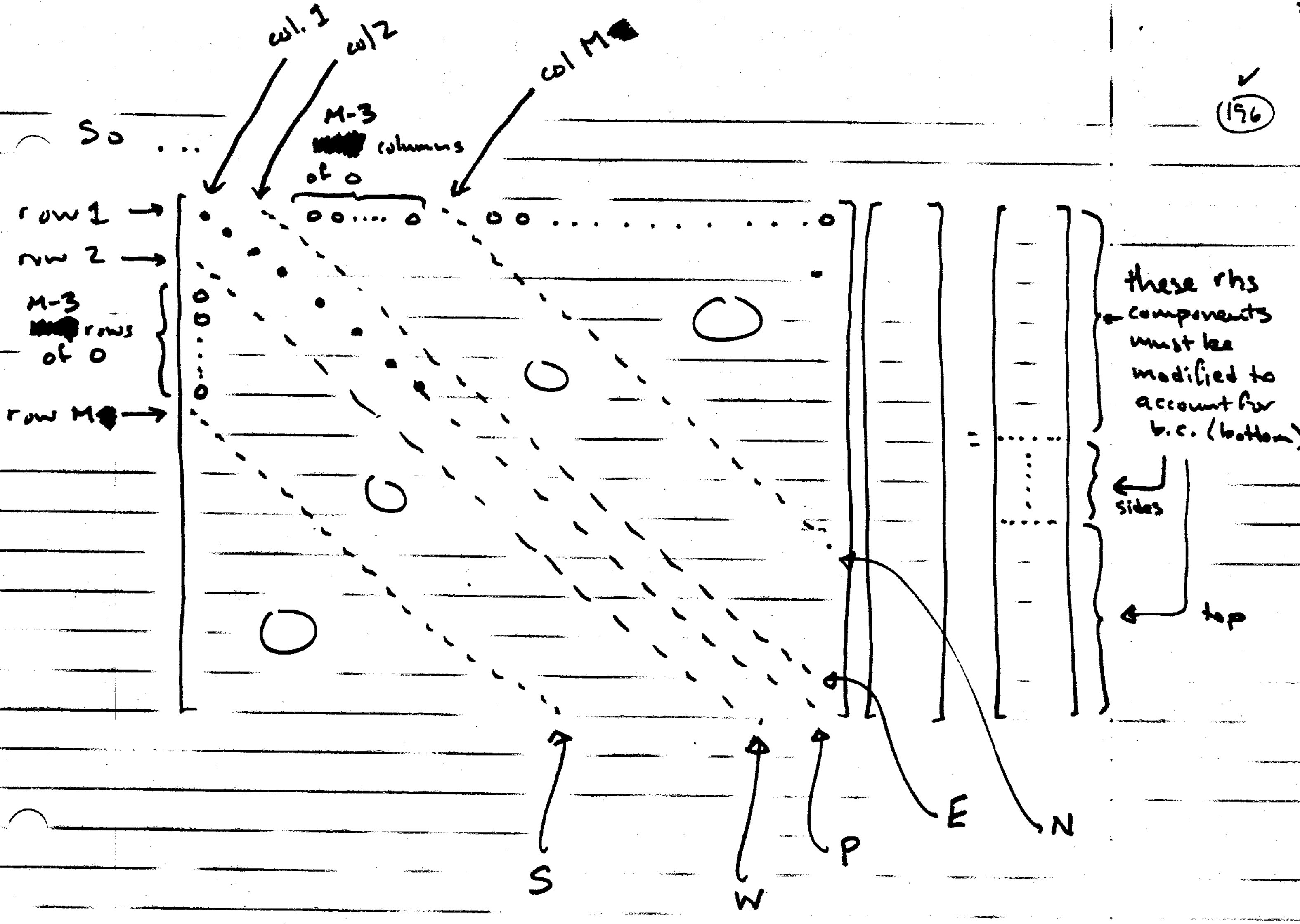
First, lets examine what a generic interior point looks like in terms of the matrix



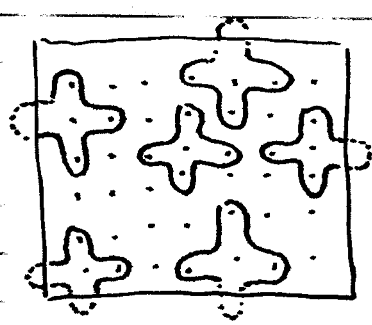


So this will be the general pattern in the matrix for points away from the boundary.

So . . .

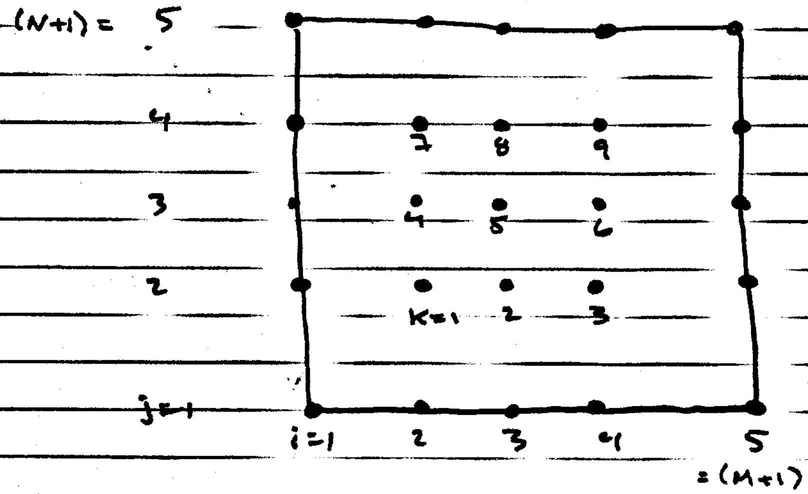


Note that in the W and E diagonals there will periodically be boundary issues to deal with as the stencil reaches the left/right boundaries.



(r=1) ✓ (197)

Consider the small grid with $\Delta x = \Delta y$ and $M = N = 4$



for $\begin{cases} \nabla^2 u = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$

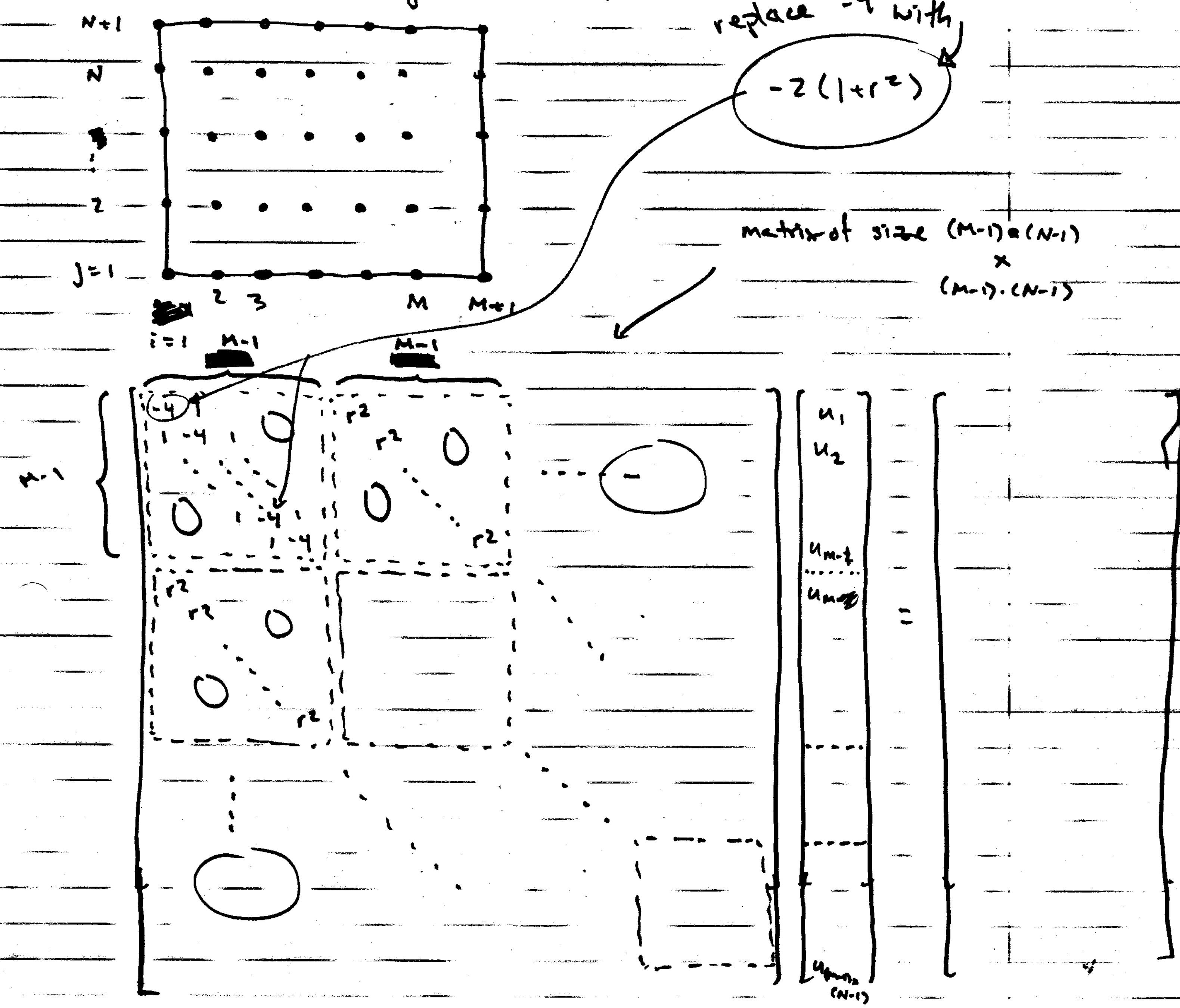
write out the matrix

$$
\begin{array}{c|c}
 & \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} &
\begin{bmatrix}
-4 & 1 & & 1 & & & & & \\
1 & -4 & 1 & & 1 & & & & \\
 & 1 & -4 & & & 1 & & & \\
1 & & & -4 & 1 & & 1 & & \\
 & 1 & & 1 & -4 & 1 & & 1 & \\
 & & 1 & & 1 & -4 & & & 1 \\
 & & & 1 & & & -4 & 1 & \\
 & & & & 1 & & 1 & -4 & 1 \\
 & & & & & 1 & & 1 & -4
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix}
=
\begin{bmatrix}
\Delta x^2 f_{2,2} - \phi_{2,1} - \phi_{1,2} \\
\Delta x^2 f_{3,2} - \phi_{3,1} \\
\Delta x^2 f_{4,2} - \phi_{5,2} - \phi_{4,1} \\
\Delta x^2 f_{2,3} - \phi_{1,3} \\
\Delta x^2 f_{3,3} \\
\Delta x^2 f_{4,3} - \phi_{5,3} \\
\Delta x^2 f_{2,4} - \phi_{2,5} - \phi_{1,4} \\
\Delta x^2 f_{3,4} - \phi_{3,5} \\
\Delta x^2 f_{4,4} - \phi_{5,4} - \phi_{4,5}
\end{bmatrix}
\end{array}
$$

matrix of size $(M-1)\cdot(N-1) \times (M-1)\cdot(N-1)$

9×9

Back to the more general case, ...



There will be $N-1$ diagonal blocks of size $(M-1) \times (M-1)$, etc. [see also Iserles, p.237 + Section 11.1 (Banded systems)]

Of course, the details of the RHS need to be worked out.

Comment

- the block pattern that appears is nice and can simplify 'small scale' programming ...

  e.g. in Matlab

$$A = [\; B11 \;\; B12 \;\; B13 \; \ldots \; ; \quad B21 \;\; B22 \;\; B23 \; \ldots \; ; \quad \ldots\ldots \quad B \;]$$

see diag kron ⋮

where
$$B11 = \begin{bmatrix} 4 & -1 & & \\ -1 & 4 & & 0 \\ & & \ddots & \\ & 0 & & 4 \end{bmatrix} \quad m\text{-}2 \times m\text{-}2 \text{ matrix}$$

but also see commands 'diag', etc. for other easy ways to fill a sparse matrix.

Also, the command 'spy(A)' will display the sparsity pattern of your matrix (shows dots for nonzero entries).

- for larger systems more efficient ways of storing (+ solving) the matrix should be used. (e.g. don't store the whole matrix but just the diagonals ...) [See Chapter 11 - Banded Systems 11.1]

What's the pattern for filling in the rhs?

(relate to blocks...)

$$
\text{rhs} = [redacted]\,\Delta x^2
\overset{\text{f-terms}}{\begin{bmatrix} f_{2,2} \\ f_{3,2} \\ \vdots \\ f_{M,2} \\ \hdashline f_{2,3} \\ f_{3,3} \\ \vdots \\ f_{M,3} \\ \hdashline \vdots \\ \hdashline f_{2,N} \\ f_{3,N} \\ \vdots \\ f_{M,N} \end{bmatrix}}
-
\overset{\phi \text{ boundary terms}}{\begin{bmatrix} r^2\phi_{2,1} + \phi_{1,2} \\ r^2\phi_{3,1} \\ \vdots \\ r^2\phi_{M,1} + \phi_{M+1,2} \\ \hdashline 0 + \phi_{1,3} \\ 0 \\ \vdots \\ 0 \\ 0 + \phi_{M+1,3} \\ \hdashline \vdots \\ \hdashline r^2\phi_{2,N+1} + \phi_{1,N} \\ r^2\phi_{3,N+1} \\ \vdots \\ r^2\phi_{M,N+1} + \phi_{M+1,N} \end{bmatrix}}
$$

→ bottom ($y=0$)

→ top ($y=1$)

→ left + right sides

So the top + bottom boundary values contribute to the first and last "block" in [redacted] the rhs. The left + right boundary values contribute to the first and last component, [redacted] respectively, of each block.

[index with $k = 1, 2, \ldots,$ [redacted] $(M-1)\cdot(N-1)$]

i.e. for points i,j with

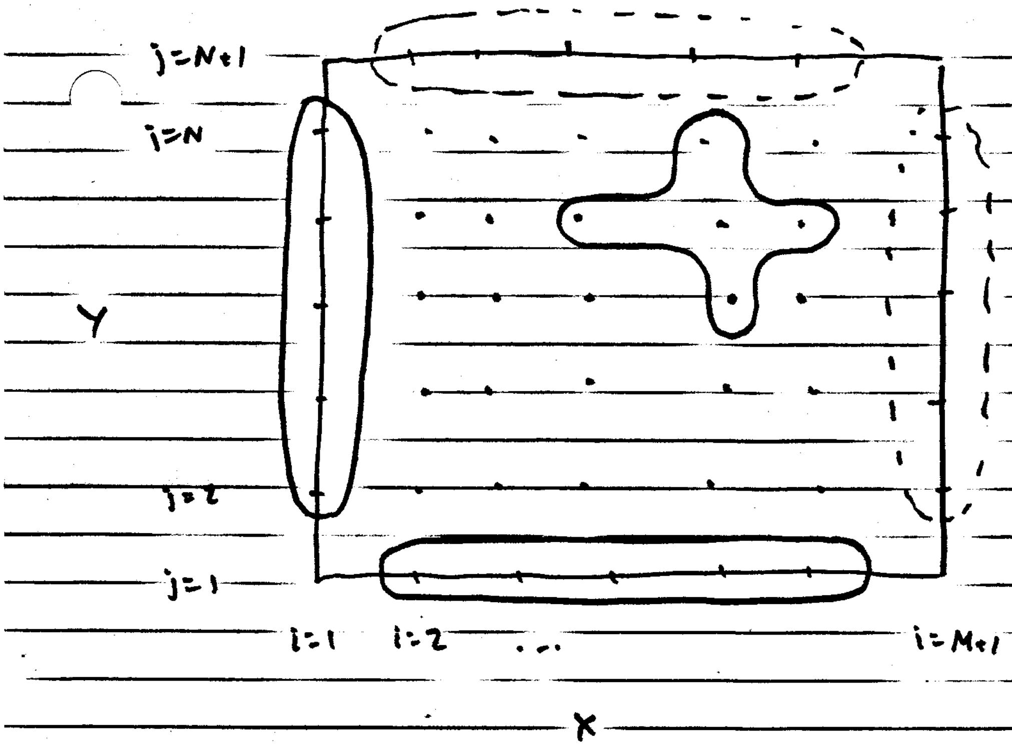
$i = 2, 3, \ldots, M$ interior pts.

and $j = 2, 3, \ldots, N$ $u_{i,j}$

corresponding unknown u_k and forcing f_k where

$$k = (j-2)(M-1) + (i-1)$$

visually, in terms of the original x-y grid...



So our basic task is to solve

$$A x = b$$

where A has a banded structure.

Before attempting to solve this linear system, what can we say to address the questions:

1. Is the linear system nonsingular?
   (so that our approximate solution exists + is unique)

   → see Lemma 1 + the corollary that follows
   (Iserles p.151-153)

2. Suppose that the solution ~~approximation~~ to $Ax=b$ is unique and exists for $\Delta x = \Delta y \to 0$. Does the numerical solution converge to the exact solution of
   $\nabla^2 u = f \qquad (x,y) \in \Omega$
   $u = \phi \qquad (x,y) \in \partial\Omega$
   If so, what is the magnitude of the error?

   → see Thm 8.2 (Iserles, p.154)

3. Are there [illegible] good ways to solve $Ax=b$ ~~to take advantage of the banded structure?~~

   → see Chapter 11

   – For now we will not pursue any "fancy" methods.

$m+2 = M+1$

$m = M-1$

Lemma 8.1

The matrix A (arising from discretizing $\nabla^2 u$ with the five-point stencil) is symmetric and the set of eigenvalues is

$$\sigma(A) = \left\{ \lambda_{\alpha,\beta} : \alpha,\beta = 2,3,\ldots,M \right\}$$

where

$$\lambda_{\alpha,\beta} = -4\left\{ \sin^2\left(\frac{(\alpha-1)\pi}{2M}\right) + \sin^2\left(\frac{(\beta-1)\pi}{2M}\right)\right\}$$

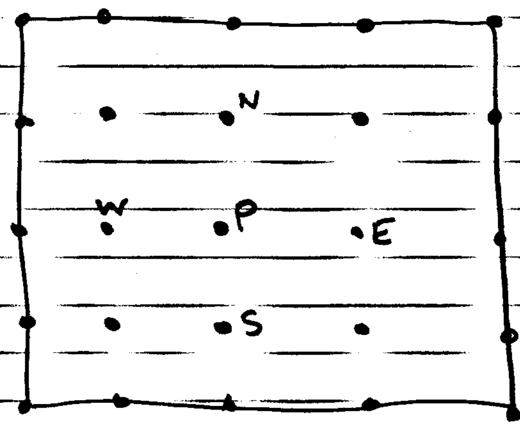
$$\alpha,\beta = 2,3,\ldots,M$$

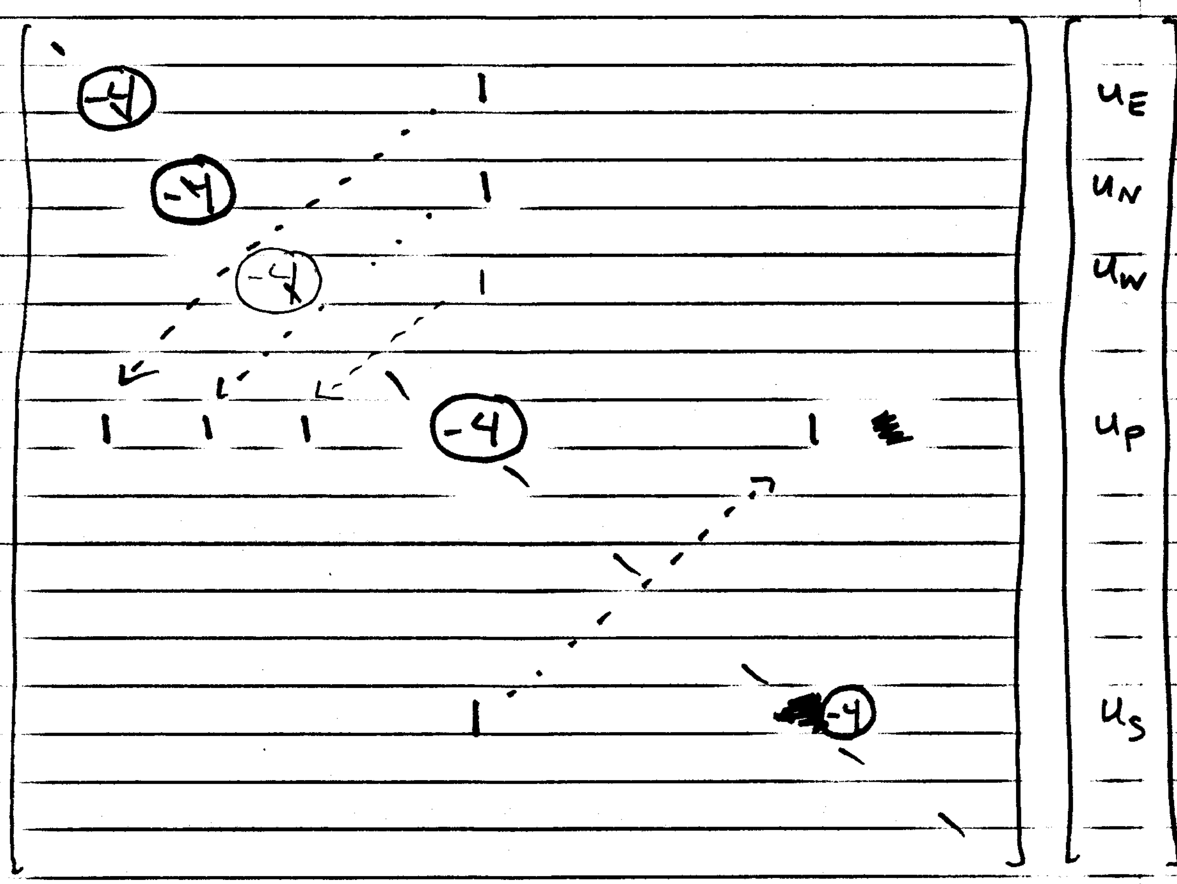
Comment:

- That A is symmetric in our derivation (with specific choice of ordering of interior points) is fairly clear from our construction. It turns out that A is symmetric for any ordering (although some orderings are better than others for computational reasons ... see ch. 11).

To see this (rather than to do the proof – see Iserles, p. 152)

Suppose the interior pts $u_{i,j}$ and their corresponding equations are filled in the matrix in an arbitrary order





- diagonal entries are always -4.

• but P is south's north
• also P is north's south
• P is east's west
• ~~P is west's east~~

The banded structure may not be so nice and the bandwidth could be large

Now lets think about the eigenvalues

(note: all eigenvalues of a symmetric matrix are real).

We'll demonstrate the eigenvalue form by showing that

$$v_{i,j} = \sin\left(\frac{(i-1)(\alpha-1)\pi}{M}\right)\sin\left(\frac{(j-1)(\beta-1)\pi}{M}\right) \qquad i,j = 1,2,\ldots,M+1$$

for given $\alpha, \beta \in \{2,3,\ldots,M\}$ satisfies

$$v_{1,j} = 0 \;;\; v_{N+1,j} = 0 \;;\; v_{i,1} = 0 \;;\; v_{i,N+1} = 0$$

and

$$v_{i-1,j} + v_{i+1,j} + v_{i,j-1} + v_{i,j+1} - 4v_{i,j} = \lambda_{\alpha,\beta}\, v_{i,j} \qquad (\text{i.e. } Ax = \lambda x)$$

for $i,j = 2,\ldots,M$ (interior points)

Observe that

$$\begin{aligned}
A \equiv\ & v_{i-1,j} + v_{i+1,j} + v_{i,j-1} + v_{i,j+1} - 4v_{i,j} \\
&= \sin\left(\frac{(i-2)(\alpha-1)\pi}{M}\right)\sin\left(\frac{(j-1)(\beta-1)\pi}{M}\right) \\
&\quad + \sin\left(\frac{i(\alpha-1)\pi}{M}\right)\sin\left(\frac{(j-1)(\beta-1)\pi}{M}\right) \\
&\quad + \sin\left(\frac{(i-1)(\alpha-1)\pi}{M}\right)\sin\left(\frac{(j-2)(\beta-1)\pi}{M}\right) \\
&\quad + \sin\left(\frac{(i-1)(\alpha-1)\pi}{M}\right)\sin\left(\frac{j(\beta-1)\pi}{M}\right) \\
&\quad - 4\sin\left(\frac{(i-1)(\alpha-1)\pi}{M}\right)\sin\left(\frac{(j-1)(\beta-1)\pi}{M}\right)
\end{aligned}$$

$$A = \left[\sin\left(\frac{(i-2)(\alpha-1)\pi}{M}\right) + \sin\left(\frac{i(\alpha-1)\pi}{M}\right)\right]\sin\left(\frac{(j-1)(\beta-1)\pi}{M}\right)$$

$$+\left[\sin\left(\frac{(j-2)(\beta-1)\pi}{M}\right) + \sin\left(\frac{j(\beta-1)\pi}{M}\right)\right]\sin\left(\frac{(i-1)(\alpha-1)\pi}{M}\right)$$

$$-4\sin\left(\frac{(i-1)(\alpha-1)\pi}{M}\right)\sin\left(\frac{(j-1)(\beta-1)\pi}{M}\right)$$

Now use trig identity

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

with $A = (i-1)\dfrac{(\alpha-1)\pi}{M}$ $\quad B = (-1)\dfrac{(\alpha-1)\pi}{M}$

or $A = (j-1)\dfrac{(\beta-1)\pi}{M}$ $\quad B = (-1)\dfrac{(\beta-1)\pi}{M}$

$$A = 2\sin\left(\frac{(i-1)(\alpha-1)\pi}{M}\right)\cos\left(\frac{(\alpha-1)\pi}{M}\right)\sin\left(\frac{(j-1)(\beta-1)\pi}{M}\right)$$

$$+2\sin\left(\frac{(j-1)(\beta-1)\pi}{M}\right)\cos\left(\frac{(\beta-1)\pi}{M}\right)\sin\left(\frac{(i-1)(\alpha-1)\pi}{M}\right)$$

$$-4\sin\left(\frac{(i-1)(\alpha-1)\pi}{M}\right)\sin\left(\frac{(j-1)(\beta-1)\pi}{M}\right)$$

$$= -2\left[2 - \cos\left(\frac{(\alpha-1)\pi}{M}\right) - \cos\left(\frac{(\beta-1)\pi}{M}\right)\right]\sin\left(\frac{(i-1)(\alpha-1)\pi}{M}\right)\sin\left(\frac{(j-1)(\beta-1)\pi}{M}\right)$$

so $\lambda = -2\left[2 - \cos\left(\frac{(\alpha-1)\pi}{M}\right) - \cos\left(\frac{(\beta-1)\pi}{M}\right)\right] v_{i,j}$

That is, the eigenvalues can be identified as

$$\lambda_{\alpha,\beta} = -2\left[2 - \cos\left(\frac{(\alpha-1)\pi}{M}\right) - \cos\left(\frac{(\beta-1)\pi}{M}\right)\right]$$

using

$$\frac{1-\cos\theta}{2} = \sin^2\left(\frac{\theta}{2}\right)$$

gives

$$\lambda_{\alpha,\beta} = -2\left[2\sin^2\left(\frac{(\alpha-1)\pi}{2M}\right) + 2\sin^2\left(\frac{(\beta-1)\pi}{2M}\right)\right]$$

$$\boxed{\lambda_{\alpha,\beta} = -4\left[\sin^2\left(\frac{(\alpha-1)\pi}{2M}\right) + \sin^2\left(\frac{(\beta-1)\pi}{2M}\right)\right] \qquad \alpha,\beta = 2,3,\ldots,M}$$

Noting that these eigenvalues are all negative (nonzero!) proves the Corollary

Corollary

The matrix A is negative definite and, [redacted] consequently, nonsingular.

Here's a theorem related to the convergence of the five-pt. stencil approximation to the exact solution.

Ω = unit square.

Thm 8.2 (sol. of $\nabla^2 u = f$ in Ω, $u = \phi$ on $\partial\Omega$)

For a sufficiently smooth f and ϕ (b.c.) there exists a number $c > 0$, independent of Δx, such that

$$\|\vec{e}\| \le c(\Delta x)^2 \quad \text{as } \Delta x \to 0.$$

Here $\vec{e} = \vec{u}_{exact} - \vec{u}_{5pt}$

$\vec{u}_{5pt}$: approx. sol.

with $e_{i,j} = u^{exact}_{i,j} - u_{5pt\ i,j}$

$u^{exact}_{i,j}$: exact sol.; $u_{5pt\ i,j}$: approx sol. — ordered in the same way

for $i = 1, 2, \ldots, M+1$ $\quad \Delta x = \Delta y = \dfrac{1}{M}$
$j = 1, 2, \ldots, M+1$

<u>Proof</u>:

See Iserles, p. 154-155.

Comment: Iserles has some discussion of how these ideas extend to other geometries in which internal pts, boundary pts and "near boundary" pts. arise.