Ch. 2 Multistep Methods

Key Points:

- Basic Idea of Multistep Methods
  - use of previous info...
- Connection of method to interpolation theory
- Brief review of interpolation theory (monomial, Newtonian, Lagrangian bases)
- Adams Methods — interpolate $\vec{y}'$ (or $\vec{f}$)
  - Adams Bashforth (4^{th} order explicit)
  - Adams Bashforth (s-step) general
- General theory for multistep methods
  - Thm 2.1 - order of multistep
  - Thm 2.2 - convergence of multistep
    - root condition
- Backward Differentiation Formulae (interpolate $\vec{y}$) (e.g. Backward Euler, ...)

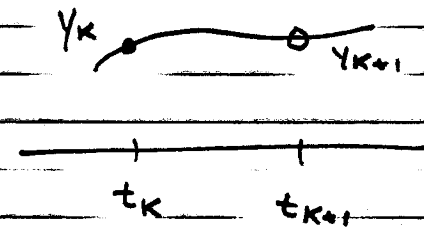
Multistep Methods (for solving $\vec{y}\,' = \vec{f}(t,\vec{y})$ $\quad \vec{y}(t_0)=\vec{y}_0$)

General idea: these use information at more than one previous point to advance to the next point (unlike Euler, Heun, Trapezoid, ...)

Recall:

Euler: $\vec{y}_{k+1} = \vec{y}_k + h\vec{f}(t_k, \vec{y}_k)$

here only ∃ information at the previous time [redacted] value $\vec{y}_k$ is required to evaluate the formula for $\vec{y}_{k+1}$. (i.e. this is a single-step method).



An example of a multistep method is a method known as Adams-Bashforth. Here, we use

4th order explicit

$$\underset{k+4}{\vec{y}_{k+1}} = \underset{k+3}{\vec{y}_k} + \frac{h}{24}\left(55\,\underset{k+3}{\vec{y}\,'_k} - 59\,\underset{k+2}{\vec{y}\,'_{k-1}} + 37\,\underset{k+1}{\vec{y}\,'_{k-2}} - 9\,\underset{k}{\vec{y}\,'_{k-3}}\right) \qquad s=4$$

where $\vec{y}\,'_k = \vec{f}(t_k, \vec{y}_k)$

$y'_{k-1} = \vec{f}(t_{k-1}, \vec{y}_{k-1})$

$\vec{y}\,'_{k-2} = \vec{f}(t_{k-2}, \vec{y}_{k-2})$

$\vec{y}\,'_{k-3} = \vec{f}(t_{k-3}, \vec{y}_{k-3})$

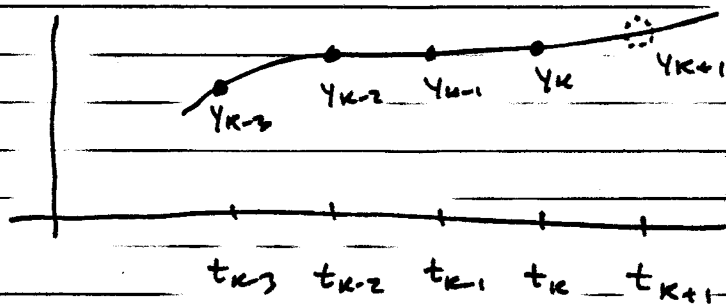
don't get too used to this notation... we are shifting the k's to something else

$k=3,4,3,5,\ldots$

(need something else for $k=0,1,2$

So in order to obtain $\vec{Y}_{K+1}$ we need to know

$\vec{Y}_K, \vec{Y}_{K-1}, \vec{Y}_{K-2}, \vec{Y}_{K-3}$ (i.e. 4 previously calculated values)



Comments:

- since we have an initial condition at only one time one might ask how we "start" this method (not "self-starting"). We need some help to get up to speed (Maybe RK?)

- You may also ask if we are even allowed to do this (specify the solution at multiple previous points) [see comment in text, p. 19 after (2.1)].

- How does one come up with such a scheme? (the numbers 55, -59, 37, -9 are not arbitrarily selected).

The general form of multistep methods we consider [redacted] is

= 1 generally, but will still write a_s in some cases...

$$a_0 \vec{y}_k + a_1 \vec{y}_{k+1} + a_2 \vec{y}_{k+2} + \dots + a_s \vec{y}_{k+s}$$

$$= h\Big[b_0 \vec{f}(t_k, \vec{y}_k) + b_1 \vec{f}(t_{k+1}, \vec{y}_{k+1}) + b_2 \vec{f}(t_{k+2}, \vec{y}_{k+2}) + \dots + b_s \vec{f}(t_{k+s}, \vec{y}_{k+s}) \Big]$$

where $s \geq 1$ is an integer. $k = 0, 1, 2, \dots$

(see also equation 2.8, p. 21 in Iserles)

Comments:

- This equation is interpreted as determining $\vec{y}_{k+s}$ given that $\vec{y}_k, \vec{y}_{k+1}, \vec{y}_{k+2}, \dots, \vec{y}_{k+s-1}$ are already known (somehow)
- It is conventional to [redacted] set $a_s = 1$
- The method is explicit if $b_s = 0$.
- The method is implicit if $b_s \neq 0$.
- In the 4th order explicit Adams-Bashforth scheme note that $s =$ [redacted]4 (and we used a shifted notation on the k-values)
- The coefficients a_j, b_j are determined by polynomial interpolation as explained below in more detail

Adams Methods: interpolate f (i.e. $\vec{y}'$)

(BDF) Backward Differentiation Formula Methods: interpolate $\vec{y}$.

See Iserles (eq. 2.5) $\rightarrow \underbrace{\vec{Y}_{k+s} = \vec{Y}_{k+s-1} + h\left[b_0 \vec{f}(t_k,\vec{Y}_k) + b_1 \vec{f}(t_{k+1},\vec{Y}_{k+1}) + \dots + b_{s-1}\vec{f}(t_{k+s-1},\vec{Y}_{k+s-1})\right]}_{\text{s-step Adams-Bashforth}}$ ✓

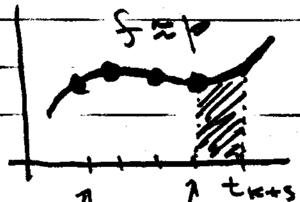
Adams Methods

$\boxed{\text{Interpolate } \vec{f}(t,\vec{y})}$ at s previous points so

$$\vec{p}(t) = \vec{f}(t,\vec{y}) \quad \text{at} \quad \begin{matrix} t_k, t_{k+1}, t_{k+2}, \dots, t_{k+s-1} \\ \underbrace{\vec{Y}_k, \vec{Y}_{k+1}, \vec{Y}_{k+2}, \dots, Y_{k+s-1}}_{s \text{ points}} \end{matrix}$$

where $\vec{p}(t)$ is a polynomial of degree $s-1$

Then, the scheme is found by integrating the polynomial (essentially extrapolating)



$$\vec{Y}_{k+s} = \vec{Y}_{k+s-1} + \int_{t_{k+s-1}}^{t_{k+s}} \vec{f}(t,\vec{y})\,dt \approx \vec{Y}_k + \int_{t_{k+s-1}}^{t_{k+s}} \vec{p}(t)\,dt$$

(note $a_0 = a_1 = \dots = a_{s-2} = 0$; $a_{s-1} = -1$) ; $b_0, b_1, b_2, \dots, b_{s-1}$ determined by the interpolation, $b_s = 0$

(see also Iserles, eq. (2.5)

Backward Difference Formula (BDF) Methods

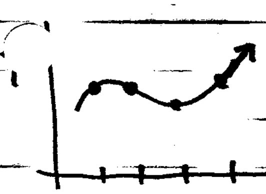
$\boxed{\text{Interpolate } \vec{Y}}$ at s previous points (i.e. $\vec{Y}_k, \vec{Y}_{k+1}, \dots, \vec{Y}_{k+s}$)
+ 1 current point

so

$$\vec{g}(t) = \vec{Y}(t) \quad \text{at} \quad \begin{matrix} t_k, t_{k+1}, t_{k+2}, \dots, t_{k+s} \\ Y_k, Y_{k+1}, Y_{k+2}, \dots, Y_{k+s} \end{matrix}$$

where $\vec{g}(t)$ is a polynomial of degree s.

Then the scheme is found by differentiating $\vec{g}(t)$ and setting it equal to $f(t_{k+s}, \vec{Y}_{k+s})$ at t_{k+s} to get $\vec{Y}_{k+s}$



$$\vec{g}\,'(t_{k+s}) = \vec{f}(t_{k+s}, \vec{Y}_{k+s})$$

Some examples

EX 1 (explicit 2step method) - Adams-type

$$y_{k+2} = y_{k+1} + \int_{t_{k+1}}^{t_{k+2}} p(t)\,dt$$

where $p(t)$ interpolates points $(t_k, \overbrace{f(t_k, y_k)}^{\vec{y}'_k})$

$(t_{k+1}, \underbrace{f(t_{k+1}, y_{k+1})}_{\vec{y}'_{k+1}})$

$$p(t) = A + B(t - t_k)$$

by inspection

$$= \underbrace{y'_k}_{A} + \underbrace{\frac{1}{h}(y'_{k+1} - y'_k)}_{B}(t - t_k)$$

$$\int_{t_{k+1}}^{t_{k+2}} p(t)\,dt = A(t_{k+2} - t_{k+1}) + \frac{1}{2}B(t - t_k)^2 \Big|_{t_{k+1}}^{t_{k+2}}$$

$$= A \cdot h + \frac{1}{2}B\left[(2h)^2 - (h)^2\right]$$

$$= Ah + \frac{3}{2}h^2 B$$

So

$$y_{k+2} = y_{k+1} + Ah + \frac{3}{2}h^2 B$$

$$= y_{k+1} + y'_k h + \frac{3}{2}h^2\left(\frac{1}{h}(y'_{k+1} - y'_k)\right)$$

$$y_{k+2} = y_{k+1} + \frac{3}{2}h\,y'_{k+1} - \frac{1}{2}y'_k$$

$$\boxed{y_{k+2} = y_{k+1} + h\left[\frac{3}{2}f(t_{k+1}, y_{k+1}) - \frac{1}{2}f(t_k, y_k)\right]}$$

(see also eq. 2.6, p.20 in Iserles)

EX 2 (4th order explicit Adams-Bashforth) $(s=4)$

$$\vec{Y}_{K+4} = \vec{Y}_{K+3} + \int_{t_{K+3}}^{t_{K+4}} \vec{p}(t)\, dt$$

where

$\vec{p}(t)$ = polynomial that interpolates (cubic) $(t_K, \vec{Y}'_K)$, $(t_{K+1}, \vec{Y}'_{K+1})$, $(t_{K+2}, \vec{Y}'_{K+2})$, $(t_{K+3}, \vec{Y}'_{K+3})$

e.g. Newton basis

$$\vec{p}(t) = \underbrace{A + B(t-t_K) + C(t-t_K)(t-t_{K+1}) + D(t-t_K)(t-t_{K+1})(t-t_{K+2})}$$

after some work A, B, C, D come out in terms of $\vec{Y}'_K, \vec{Y}'_{K+1}, \vec{Y}'_{K+2}, \vec{Y}'_{K+3}$

then insert $p(t)$ into above integral to obtain scheme.

... tedious but doable...

An alternate way to look at this and derive the coefficients is to recognize this scheme as having the form

$$\vec{Y}_{K+4} = \vec{Y}_{K+3} + h\left[\beta_0 \vec{Y}'_K + \beta_1 \vec{Y}'_{K+1} + \beta_2 \vec{Y}'_{K+2} + \beta_3 \vec{Y}'_{K+3}\right]$$

and requiring that this formula be exact if

$y' = 1, t, t^2$, or t^3 (then p would coincide exactly with $\vec{Y}'$)

(so $y = t + c_0, \frac{1}{2}t^2 + c_0, \frac{1}{3}t^3 + c_0, \frac{1}{4}t^4 + c_0$)

$$\begin{cases} y' = 1 \\ \underline{y = t + C_0} \end{cases}$$

$$t_{k+4} = t_{k+3} + h\left[\beta_0 + \beta_1 + \beta_2 + \beta_3\right] \quad *$$

$$\begin{cases} y' = t \\ \underline{y = \frac{1}{2}t^2 + C_0} \end{cases}$$

$$\frac{1}{2}(t_{k+4})^2 = \frac{1}{2}(t_{k+3})^2 + h\left[\beta_0 t_k + \beta_1 t_{k+1} + \beta_2 t_{k+2} + \beta_3 t_{k+3}\right] \quad *$$

$$\begin{cases} y' = t^2 \\ \underline{y = \frac{1}{3}t^3 + C_0} \end{cases}$$

$$\frac{1}{3}(t_{k+4})^3 = \frac{1}{3}(t_{k+3})^3 + h\left[\beta_0 t_k^2 + \beta_1 t_{k+1}^2 + \beta_2 t_{k+2}^2 + \beta_3 t_{k+3}^2\right] \quad *$$

$$\begin{cases} y' = t^3 \\ \underline{y = \frac{1}{4}t^4 + C_0} \end{cases}$$

$$\frac{1}{4}(t_{k+4})^4 = \frac{1}{4}(t_{k+4})^4 + h\left[\beta_0 t_k^3 + \beta_1 t_{k+1}^3 + \beta_2 t_{k+2}^3 + \beta_3 t_{k+3}^3\right] \quad *$$

Note that the four equations (*) must hold for any values of t so let's choose them conveniently,

$$t_k = 0$$
$$t_{k+1} = 1 \qquad h = 1.$$
$$t_{k+2} = 2$$
$$t_{k+3} = 3$$
$$t_{k+4} = 4$$

So...

$$4 = 3 + \beta_0 + \beta_1 + \beta_2 + \beta_3$$

$$16 = 9 + 2[0 + \beta_1 + 2\beta_2 + 3\beta_3]$$

$$64 = 27 + 3[0 + \beta_1 + 4\beta_2 + 9\beta_3]$$

$$256 = 81 + 4[0 + \beta_1 + 8\beta_2 + 27\beta_3]$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & 1 & 8 & 27 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 7/2 \\ 37/3 \\ 175/4 \end{bmatrix}$$

solve $\Rightarrow$ $\boxed{\beta_0 = -\frac{9}{24} \quad \beta_1 = \frac{37}{24} \quad \beta_2 = -\frac{59}{24} \quad \beta_3 = \frac{55}{24}}$

So the scheme is

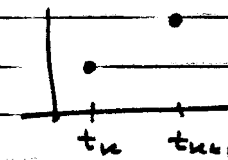
$$\boxed{y_{k+4} = y_{k+3} + \frac{h}{24}\left[-9y'_k + 37y'_{k+1} - 59y'_{k+2} + 55y'_{k+3}\right]}$$

Adams-Bashforth (4th order explicit)

compare form, p. 22 in notes...

EX 3 (BDF with $s=1$ ... Backward Euler)

$$\vec{g}'(t_{k+1}) = \vec{f}(t_{k+1}, y_{k+1})$$



where $\vec{g}$ interpolates $(t_{k+1}, \vec{y}_{k+1})$ and $(t_k, \vec{y}_k)$

This is just a linear function

$$\vec{g}(t) = \cancel{y}_k + \frac{(\vec{y}_{k+1} - \vec{y}_k)}{h}(t - t_k)$$

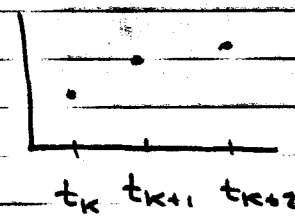
so

$$\vec{g}'(t) = \frac{\vec{y}_{k+1} - \vec{y}_k}{h}$$

and the scheme is ...

$$\boxed{\frac{\vec{y}_{k+1} - \vec{y}_k}{h} = f(t_{k+1}, \vec{y}_{k+1})}$$ i.e. Backward Euler.

EX 4 (BDF with $s=2$)



Here, $\vec{g}(t)$ interpolates (t_k, y_k), (t_{k+1}, y_{k+1}), (t_{k+2}, y_{k+2})

... a little work shows

$$g(t) = A + B(t - t_k) + C(t - t_k)(t - t_{k+1})$$

where

$$A = y_k, \quad B = \frac{y_{k+1} - y_k}{h}, \quad C = \frac{y_{k+2} - 2y_{k+1} + y_k}{2h^2}$$

So that ...

$$\boxed{\vec{y}_{k+2} - \frac{4}{3}\vec{y}_{k+1} + \frac{1}{3}\vec{y}_k = \frac{2}{3}h\vec{f}(t_{k+2}, \vec{y}_{k+2})}$$

We'd like to now address the issue of "order of the method".
As we did for the previous ~~steps~~ chapter (see def. p.8 in Iserles) we can characterize the order of the method by inserting the exact solution into the formula for the scheme. In the present context, then, we have an "order p" method if

$$\boxed{\begin{aligned} &a_0 \vec{y}(t_k) + a_1 \vec{y}(t_{k+1}) + \dots + a_s\vec{y}(t_{k+s}) \\ &\quad - h\left[b_0 \vec{f}(t_k, \vec{y}(t_k)) + b_1 \vec{f}(t_{k+1}, \vec{y}(t_{k+1})) + \dots + b_s \vec{f}(t_{k+s}, \vec{y}(t_{k+s}))\right] \\ &= O(h^{p+1}) \qquad\qquad t_{k+1} = t_k + h, \dots, t_{k+s} = t_k + s\cdot h \end{aligned}}$$

└ Multistep Method of order p, as $h \to 0$.

Comment:

- previously, we had taken the approach of writing

$$\vec{f}(t_{k+1}, \vec{y}(t_{k+1})) = \vec{y}'(t_{k+1}) \approx \vec{y}'(t_k) + h\vec{y}''(t_k) + \tfrac{1}{2}h^2\vec{y}'''(t_k) + \dots$$

etc.

(i.e. Taylor Expand $\vec{y}$ and $\vec{y}'$ everywhere and see what remains...)

- we'll do the same here but we'll introduce some notation, as in Iserles, to help streamline things...

... the result will be Thm 2.1 (Iserles, p.22)

Let $$\psi(t,\vec{y}) \equiv \sum_{m=0}^{s} a_m \vec{y}(t+mh) - h \sum_{m=0}^{s} b_m \vec{y}'(t+mh)$$

Assume $\vec{y}$ is analytic and that its radius of convergence is greater than the maximum size of mh — i.e. sh.

So $$\vec{y}(t+mh) = \sum_{k=0}^{\infty} \frac{1}{k!} \vec{y}^{(k)}(t)(mh)^k = \vec{y}(t) + mh\,\vec{y}'(t) + \tfrac{1}{2}(mh)^2 \vec{y}''(t) + \ldots$$

and $$\vec{y}'(t+mh) = \sum_{k=0}^{\infty} \frac{1}{k!} \vec{y}^{(k+1)}(t)(mh)^k = \vec{y}'(t) + mh\,\vec{y}''(t) + \tfrac{1}{2}(mh)^2 \vec{y}'''(t) + \ldots$$

Then
$$\psi(t,\vec{y}) = \sum_{m=0}^{s} a_m \left[\sum_{k=0}^{\infty} \frac{1}{k!} \vec{y}^{(k)}(t)(mh)^k \right] - h \sum_{m=0}^{s} b_m \left[\sum_{k=0}^{\infty} \frac{1}{k!} \vec{y}^{(k+1)}(t)(mh)^k \right]$$

$$= \sum_{m=0}^{s} a_m \left[\vec{y}(t) + \sum_{k=1}^{\infty} \frac{1}{k!} \vec{y}^{(k)}(t)(mh)^k \right] - h \sum_{m=0}^{s} b_m \left[\sum_{k=0}^{\infty} \frac{1}{k!} \vec{y}^{(k+1)}(t)(mh)^k \right]$$

$$= \vec{y}(t)\left(\sum_{m=0}^{s} a_m \right) + \sum_{m=0}^{s} a_m \left[\sum_{k=1}^{\infty} \frac{1}{k!} \vec{y}^{(k)}(t)(mh)^k \right]$$
$$- h \sum_{m=0}^{s} b_m \left[\sum_{j=1}^{\infty} \frac{1}{(j-1)!} \vec{y}^{(j)}(t)(mh)^{j-1} \right]$$

$$= \vec{y}(t)\left(\sum_{m=0}^{s} a_m \right) + \sum_{k=1}^{\infty} \left(\sum_{m=0}^{s} a_m m^k \right) \frac{1}{k!} h^k \vec{y}^{(k)}(t)$$
$$- h \sum_{k=1}^{\infty} \left(\sum_{m=0}^{s} b_m m^{k-1} \right) \frac{k}{k!} \vec{y}^{(k)} h^{k-1}$$

$$\vec{\Psi}(t,\vec{y}) = \vec{y}(t)\left(\sum_{m=0}^{s} a_m\right)$$

$$+ \sum_{k=1}^{\infty} \left\{ \left(\sum_{m=0}^{s} m^k a_m\right) - k\left(\sum_{m=0}^{s} m^{k-1} b_m\right)\right\} \frac{1}{k!} h^k \vec{y}^{(k)}(t)$$

In order for this method to be order p (i.e. $\Psi \sim O(h^{p+1})$) we need

$$\boxed{\sum_{m=0}^{s} a_m = 0}$$

$$\boxed{\sum_{m=0}^{s} m^k a_m = k \sum_{m=0}^{s} m^{k-1} b_m} \quad \text{for } k=1,2,3,\dots,p$$

with

$$\boxed{\sum_{m=0}^{s} m^{p+1} a_m \neq (p+1) \sum_{m=0}^{s} m^p b_m}$$

These conditions can be expressed in terms of the polynomials

$$\rho(w) \equiv \sum_{m=0}^{s} a_m w^m$$

$$\sigma(w) \equiv \sum_{m=0}^{s} b_m w^m$$

(see book page 22 for further details...)

The resulting theorem can be expressed in terms of $\rho(w)$ and $\sigma(w)$ ...

Thm 2.1

Our generic multistep method is of order $p \geq 1$ if and only if there exists [redacted] $c \neq 0$ such that

$$\rho(w) - \sigma(w) \ln w = c(w-1)^{p+1} + O\left(|w-1|^{p+2}\right)$$

as $w \to 1$

Comments:

- The local error properties of the method can be characterized in terms of polynomials defined in the above way by the coefficients $a_0, a_1, \ldots, a_s$, $b_0, b_1, \ldots, b_s$.

A second Theorem tells us about convergence of these methods...

Thm 2.2. (The Dahlquist Equivalence Thm)

Suppose the error in the starting values $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_{s-1}$ tends to zero as $h \to 0^+$. Our generic multistep method is convergent iff it is of order $p \geq 1$ and the polynomial $\rho(w)$ has all its zeros inside the closed complex unit disc and all its zeros of unit modulus are simple. (root condition, see p. 24)

What do these results tell us for our specific examples?...

EX 1

$$y_{k+2} = y_{k+1} + h\left[\frac{3}{2}y'_{k+1} - \frac{1}{2}y'_k\right]$$

$s = 2$ $\quad a_0 = 0 \quad b_0 = -\frac{1}{2}$

$\quad a_1 = -1 \quad b_1 = 3/2$

$\quad a_2 = 1 \quad b_2 = 0$

$$\rho(w) = \sum_{m=0}^{s} a_m w^m = a_0 + a_1 w + a_2 w^2 = -w + w^2 = w(w-1)$$

$$\sigma(w) = \sum_{m=0}^{s} b_m w^m = b_0 + b_1 w + b_2 w^2 = -\frac{1}{2} + \frac{3}{2}w$$

order: $(w \to 1)$ $\quad \rho(w) - \sigma(w)\ln w = w(w-1) - \left(\frac{3}{2}w - \frac{1}{2}\right)\ln w$

Note: $\ln w = \ln(1+(w-1)) \sim (w-1) - \frac{1}{2}(w-1)^2 + \frac{1}{3}(w-1)^3 + \ldots$

so

$$\rho(w) - \sigma(w)\ln w = w(w-1) - \frac{1}{2}(3w-1)\left[(w-1) - \frac{1}{2}(w-1)^2 + \frac{1}{3}(w-1)^3 + \ldots\right]$$

$$= (w-1)\left\{w - \frac{1}{2}(3w-1)\left[1 - \frac{1}{2}(w-1) + \frac{1}{3}(w-1)^2 + \ldots\right]\right\}$$

$$= (w-1)\left\{w - \frac{1}{2}(3w-1)\left[\frac{3}{2} - \frac{1}{2}w + \frac{1}{3}(w-1)^2 + \ldots\right]\right\}$$

$$= (w-1)\left\{w - \frac{1}{4}(9w - 3w^2 - 3 + w) - \frac{1}{6}(3w-1)(w-1)^2 + \ldots\right\}$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \qquad -1 < x \leq 1$$

$$\rho(w) - \sigma(w)\ln w \sim (w-1)\left\{-\frac{1}{4}\left(-3w^2+10w-3-4w\right) - \frac{1}{6}(3w-1)(w-1)^2 + \dots\right\}$$

$$\sim (w-1)\left\{-\frac{1}{4}\left(-3w^2+6w-3\right) - \frac{1}{6}(3w-1)(w-1)^2 + \dots\right\}$$

$$\sim (w-1)\left\{\frac{3}{4}(w-1)^2 - \frac{1}{6}(3w-1)(w-1)^2 + \dots\right\}$$

$$\sim (w-1)^3\left\{\frac{3}{4} - \frac{1}{6}(3w-1) + O(w-1)\right\}$$

$$\sim \underbrace{(w-1)^3}_{\text{looks like } (w-1)^{p+1}}\left\{-\frac{1}{2}w + \frac{11}{12} + O(w-1)\right\}$$

So this method is order 2

Also note

$\rho(w) = w(w-1)$ obeys the root condition

$\Rightarrow$ method is <u>convergent</u>.

Back to Adams-Bashforth (4th order explicit)

This method is often used together with the Adams-Moulton method in a predictor—corrector scheme

<u>Adams Moulton</u> (implicit 4th order scheme)

$$\boxed{\vec{Y}_{K+3} = \vec{Y}_{K+2} + \frac{h}{24}\left[9\vec{Y}'_{K+3} + 19\vec{Y}'_{K+2} - 5\vec{Y}'_{K+1} + \vec{Y}'_K\right]}$$

<u>Adams Bashforth</u> (explicit 4th order scheme) (slightly rewritten) shift index

$$\boxed{\vec{Y}_{K+3} = \vec{Y}_{K+2} + \frac{h}{24}\left[-9Y'_{K-1} + 37Y'_K - 59Y'_{K+1} + 55Y'_{K+2}\right]}$$

<u>Idea:</u>

Predict: $\vec{Y}_{K+3}$ using AB

Evaluate: $\vec{f}(t_{K+3}, \vec{Y}_{K+3})$ to estimate $\vec{Y}'_{K+3}$ in AM

Correct: use AM to get a new $\vec{Y}_{K+3}$

Evaluate: $\vec{f}(t_{K+3}, \vec{Y}_{K+3})$ and repeat the corrector AM

can continue iterating

Note: This is similar to Heun's method

$$\begin{cases} \hat{Y}_{K+1} = Y_K + f(t_K, Y_{K+1}) \\ Y_{K+1} = Y_K + \frac{1}{2}\left[f(t_K, Y_K) + f(t_{K+1}, \hat{Y}_{K+1})\right] \end{cases}$$

i.e. Euler/Trapezoid Predictor/Corrector.

Recall that the interpolation error associated with a polynomial of degree at most $n-1$ that interpolates f n distinct points $t_1, t_2, \ldots, t_n$ (f is sufficiently smooth) is

$$f(t) - p_{n-1}(t) = \frac{f^{(n)}(\theta)}{n!} \prod_{i=1}^{n} (t - t_i) \qquad \theta \in (t_1, t_n)$$

(e.g. Heath, p. 324; Iserles A.2.2.2)

~~Note that if the points are equally-spaced~~