Ch 1. Euler's Method + Related Ideas

Key Points :

- 1st order ODE (IVP) $\vec{y}' = \vec{f}(t, \vec{y})$, $\vec{y}(t_0) = \vec{y}_0$

- Euler's Method
  - basic idea
  - various derivations (differential form, integral form)

- Simple example $\begin{cases} y' = 3y \\ y(0) = 1 \end{cases}$

- More general conditions — Lipschitz condition

- convergence proof (general case)

- error - local/global

- Definition of "Method of order p"

- Trapezoid Method (Implicit Method)
  - integral derivation
  - error
  - convergence proof

- Theta Method
  - Euler, Trapezoid, Backward Euler
  - error

- Modified Euler — Heun's Method
  - error

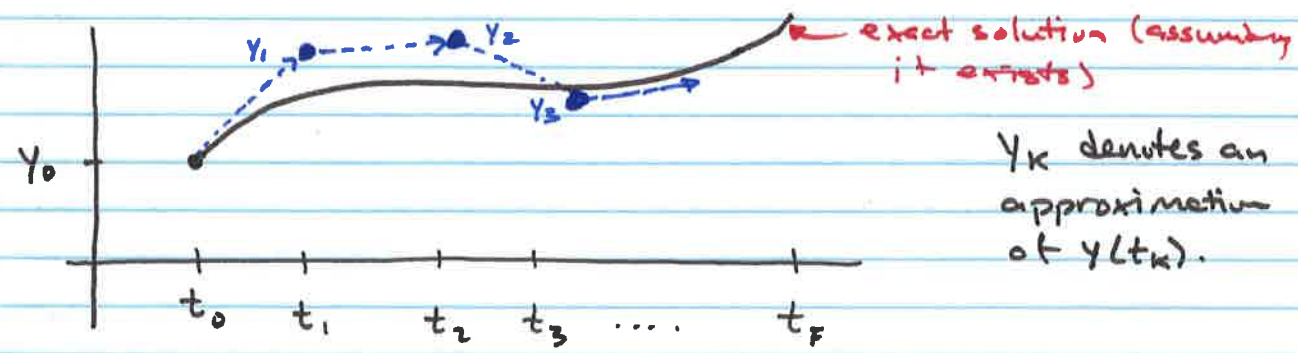
- Simpson's Method

Let's start by considering a simple one-dimensional first order initial value problem.

$$\begin{cases} \dfrac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

where f is sufficiently smooth – more on this later...

We'd like to solve this problem — find $y(t)$ — up to some final time t_F ... numerically at a discrete set of points



y_k denotes an approximation of $y(t_k)$.

Plan: use "slope" information to advance a small distance in time

$$y_1 = y_0 + (t_1 - t_0)\cdot f(t_0, y_0) \qquad \text{(gives } y_1\text{)}$$

then

$$y_2 = y_1 + (t_2 - t_1)\cdot f(t_1, y_1) \qquad \text{(gives } y_2\text{)}$$

in general

$$y_{k+1} = y_k + \underbrace{(t_{k+1} - t_k)}_{h_k} f(t_k, y_k)$$

Euler's Method $\boxed{y_{k+1} = y_k + h_k f(t_k, y_k)}$ $k = 1, 2, 3, \ldots$ start with y_0

explicit formula to "advance" in time.

- One way to view this is ... rewriting

$$\frac{y_{k+1} - y_k}{h_k} = f(t_k, y_k)$$

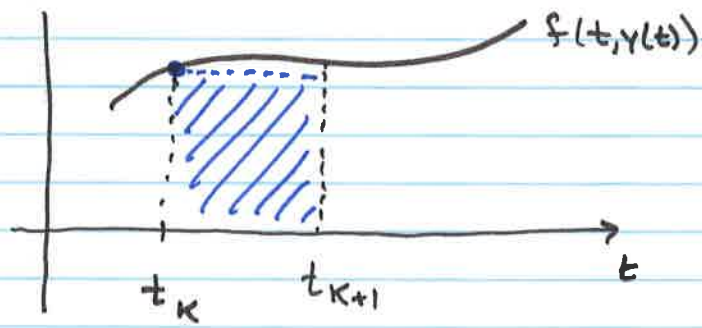
Essentially we approximate $\frac{dy}{dt} = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}$

approximate the derivative $\approx \frac{y_{k+1} - y_k}{h_k}$

where h_k is "small"

- Another way to view this is an an integral, or quadrature, problem

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(t, y)\, dt \qquad (\text{integrate } y' = f(t,y))$$



Approximate the integral (area) ... try left-endpoint approx.

$$\boxed{y_{k+1} = y_k + \underbrace{(t_{k+1} - t_k)}_{h_k} \cdot f(t_k, y_k)} \quad \text{Euler's Method}$$

We'll revisit this quadrature approach again later...

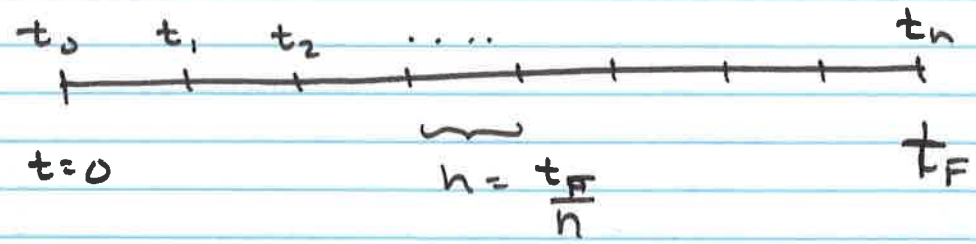
Let's examine a simple example

EXAMPLE

Solve
$$\begin{cases} y' = 3y \\ y(0) = 1 \end{cases}$$

using Euler's Method. on $0 \le t \le t_F$

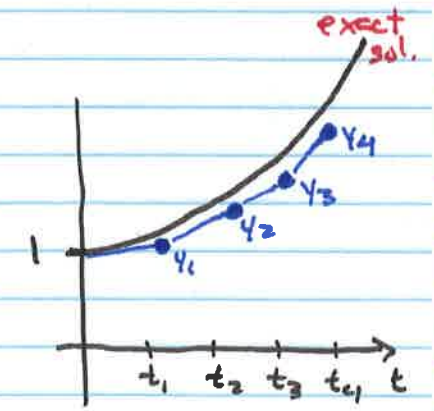
Set up a grid (uniform spacing for simplicity)



Euler

$$y_{k+1} = y_k + h f(t_k, y_k)$$

$$y_{k+1} = y_k + h \cdot 3y_k = (1+3h)y_k$$



so

$$y_1 = (1+3h)y_0 = (1+3h)$$

$$y_2 = (1+3h)y_1 = (1+3h)^2$$

$$\vdots$$

$$y_n = (1+3h)y_{n-1} = (1+3h)^n = \left(1+\frac{3t_F}{n}\right)^n$$

Note: $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$

So $\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(1+\frac{3t_F}{n}\right)^n = e^{3t_F}$

so y_n converges to the exact solution $y(t) = e^{3t}$

* normally we can't solve these explicitly but this was a nice, easy, linear problem.

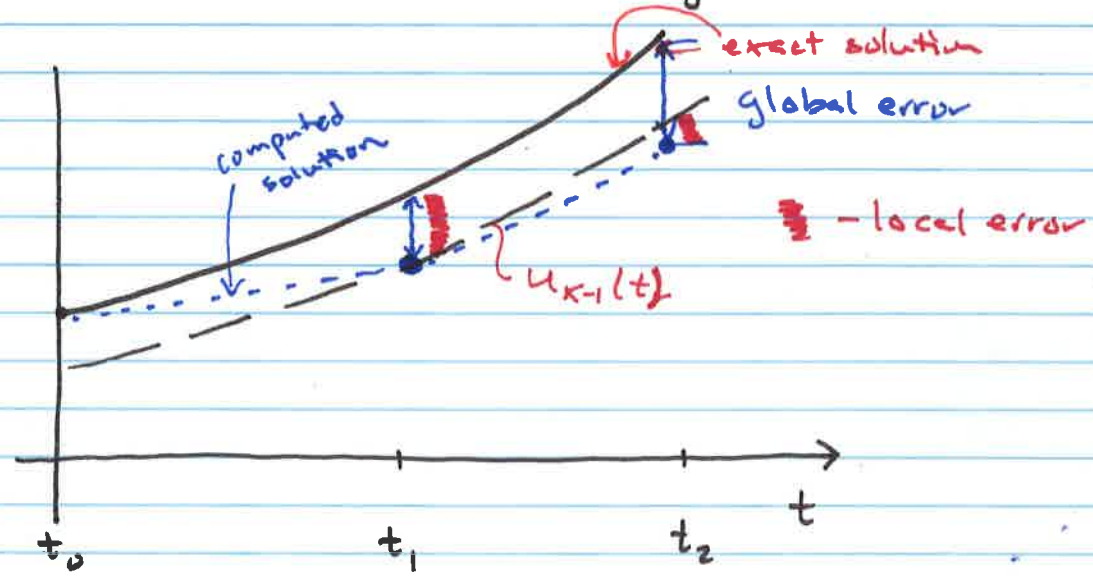
sticking to 1D still...

<u>Error in Euler's Method</u> ... two sorts of error to think about

global error $\equiv e_k = y_k - y(t_k)$

local error $\equiv \ell_k = y_k - u_{k-1}(t_k)$

$u_{k-1}(t) =$ solution of ODE passing through previous point



Taylor Expand $y(t_{k+1})$ and compare with Euler

$$y(t_{k+1}) \approx y(t_k) + y'(t_k)h + \tfrac{1}{2}h^2 y''(t_k) + \ldots$$

$$y_{k+1} = y_k + h\, f(t_k, y_k)$$

$$y(t_{k+1}) - y_{k+1} = y(t_k) - y_k + h\left[y'(t_k) - f(t_k, y_k)\right] + \tfrac{1}{2}h^2 y''(t_k) + \ldots$$

$$= y(t_k) - y_k + h\left[f(t_k, y(t_k)) - f(t_k, y_k)\right] + \tfrac{1}{2}h^2 y''(t_k) + \ldots$$

$$f(t_k, y(t_k)) \approx f(t_k, y_k) + \frac{\partial f}{\partial y}(t_k, y_k)\left(y(t_k) - y_k\right) + \dots$$

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$$y(t_{k+1}) - y_{k+1} = y(t_k) - y_k + h\left[\frac{\partial f}{\partial y}(t_k, y_k)\left(y(t_k) - y_k\right)\right] + \tfrac{1}{2}h^2 y''(t_k)$$

$$\underbrace{y(t_{k+1}) - y_{k+1}}_{\text{new global error}} = \underbrace{\left[1 + h\frac{\partial f}{\partial y}(t_k, y_k)\right]}_{\text{amplification factor}}\underbrace{\left[y(t_k) - y_k\right]}_{\text{previous global error}} + \underbrace{\tfrac{1}{2}h^2 y''(t_k)}_{\text{local error}} + \dots$$

This is the error generated if the solution at the previous step is exact.

$\ell_k \sim h^2$

Euler's method

"first order"

or

"first order accurate"

We also see the familiar (?) condition for stability of the numerical scheme ... i.e.

$$\left|1 + h\frac{\partial f}{\partial y}\right| \le 1$$

Expect that global error $\sim n \cdot h^2 = \left(\frac{T}{h}\right)h^2 \sim \underline{\underline{O(h)}}$

Let's see if we can make this a little more rigorous + generalize for

$$\begin{cases} \vec{y}\,' = \vec{f}(t, \vec{y}) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases}$$

So consider ...

$$\begin{cases} \vec{y}\,' = \vec{f}(t,\vec{y}) & t > t_0 \\ \vec{y}(t_0) = \vec{y}_0 \end{cases}$$

$\vec{y}, \vec{y}_0 \in \mathbb{R}^d$ $\quad \vec{f} : [t_0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ $\quad \mathbb{R}^d$ = Euclidean Space

For $\vec{f}$ we require that it satisfy the <u>Lipshitz Condition</u>

$$* \quad \boxed{\begin{array}{l} \|\vec{f}(t,\vec{x}) - \vec{f}(t,\vec{y})\| \leq \lambda \|\vec{x} - \vec{y}\| \\ \text{for all } \vec{x}, \vec{y} \in \mathbb{R}^d, \ t \geq t_0, \ \lambda = \text{constant indep. of } \vec{x}, \vec{y} \end{array}}$$

(i.e. there exists a $\lambda > 0$ s.t. this condition holds

<u>Note</u>: • we'll often end up requiring that $\vec{f}$ has a Taylor Series about every point $(t, \vec{y})$ [$\vec{f}$ analytic ... T. Series has a positive radius of convergence.]

• one might conceptually associate λ with $\dfrac{\partial \vec{f}}{\partial \vec{y}}$ (or some measure thereof...)

<u>Euler's Method</u> (constant step size for simplicity)

$$\boxed{\vec{y}_{k+1} = \vec{y}_k + h\,\vec{f}(t_k, \vec{y}_k)}$$

We saw for a simple 1D example that the solution to Euler's Method applied to $y' = 3y$ converged to the exact solution.

More generally, ~~convergence requires the following~~ a method is said to be <u>convergent</u> if

$$\lim_{h \to 0} \left(\max_k \| y_k - y(t_k) \| \right) = 0$$

for every ODE with Lipshitz function $\vec{f}$

(see text, p. 6 ... slightly streamlined notation)

y_k is understood to depend on h.

Let's revisit Euler's Method and show

Thm 1.1 Euler's Method is convergent.

To show this ...

Taylor Expand Exact Sol'

$$\vec{y}(t_{k+1}) \approx \vec{y}(t_k) + h\,\vec{y}'(t_k) + \tfrac{1}{2}h^2\,\vec{y}''(t_k) + O(h^3)$$

$$\vec{y}_{k+1} = \vec{y}_k + h\,\vec{f}(t_k, \vec{y}_k)$$

$$\underbrace{\vec{y}(t_{k+1}) - \vec{y}_{k+1}}_{\equiv\, \vec{e}_{k+1}} = \underbrace{\vec{y}(t_k) - \vec{y}_k}_{\vec{e}_k} + h\left[\vec{f}(t_k, \vec{y}(t_k)) - \vec{f}(t_k, \vec{y}_k)\right] + O(h^2)$$

(note: Iserles uses different sign) but no worries, norms are coming

$$\vec{e}_{k+1} = \vec{e}_k + h\left[\vec{f}(t_k, \vec{y}(t_k)) - \vec{f}(t_k, \vec{y}(t_k) - \vec{e}_k)\right] + O(h^2)$$

$\|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$

OK ✓

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Taking norms...

$$\|\vec{e}_{k+1}\| = \left\| \vec{e}_k + h\left[\vec{f}(t_k,\vec{y}(t_k)) - \vec{f}(t_k,\vec{y}(t_k)-\vec{e}_k)\right] + O(h^3)\right\|$$

triangle inequality

$$\le \|\vec{e}_k\| + h\left\|\vec{f}(t_k,\vec{y}(t_k)) - \vec{f}(t_k, \underbrace{\vec{y}(t_k)-\vec{e}_k}_{\vec{x}})\right\| + ch^2$$

Lipschitz cond.

$$\le \|\vec{e}_k\| + h\lambda\left\|\vec{y}(t_k) - (\vec{y}(t_k)-\vec{e}_k)\right\| + ch^2$$

$$= \|\vec{e}_k\| + h\lambda\|\vec{e}_k\| + ch^2$$

c is a positive constant

so

$$\boxed{\|\vec{e}_{k+1}\| \le (1+h\lambda)\|\vec{e}_k\| + ch^2}$$

For convergence, we'd like to show $\|e_k\| \to 0$ as $h \to 0$.

$$\|e_1\| \le ch^2 \qquad \text{note: } \|\vec{e}_0\| = 0$$

$$\|e_2\| \le \underbrace{(1+h\lambda)(ch^2) + (ch^2)}_{ch^2[1+(1+h\lambda)]}$$

$$\|e_3\| \le \underbrace{(1+h\lambda)\,ch^2[1+(1+h\lambda)] + ch^2}_{ch^2[1+(1+h\lambda)+(1+h\lambda)^2]}$$

$$\vdots$$

$$\|e_k\| \le ch^2\left[1+(1+h\lambda)+(1+h\lambda)^2+\dots+(1+h\lambda)^{k-1}\right]$$

but $S = 1 + p + p^2 + \dots + p^{k-1}$

$$pS = p + p^2 + p^3 + \dots + p^k$$

$$S - pS = 1 - p^k$$

$$S = \frac{1-p^k}{1-p}$$

so

$$\|\vec{e}_k\| \leq ch^2 \left[\frac{1-(1+h\lambda)^k}{1-(1+h\lambda)} \right] = ch^2 \left[\frac{1-(1+h\lambda)^k}{-h\lambda} \right]$$

$$\boxed{\|\vec{e}_k\| \leq \frac{ch}{\lambda}\left[(1+h\lambda)^k - 1\right]}$$

This proves that Euler's Method is convergent (i.e. $\|e_k\| \to 0$ as $h \to 0$) and that the global error decays as $O(h)$. (*)

since

$1 + h\lambda < e^{h\lambda}$ (e.g. $e^{h\lambda} = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \dots$)

$(1+h\lambda)^n < e^{h\lambda n}$ but $hn = t_F$ ($h = \frac{t_F}{n}$)

so $(1+h\lambda)^n < e^{\lambda t_F}$

so

$$\boxed{\|\vec{e}_k\| \leq \frac{ch}{\lambda}\left[e^{\lambda t_F} - 1\right]}$$

General Method of Order p

Consider

$$y_{k+1} = \Psi_k(\vec{f}, h, \vec{y}_0, \vec{y}_1, \ldots, \vec{y}_k)$$

single-step or multi-step method

We say this method is order p [i.e. local error $\sim O(h^{p+1})$]

if when we equate y_{k+1} with $y(t_{k+1})$ and y_k with $y(t_k)$, etc.

we find

$$y(t_{k+1}) - \Psi_k(\vec{f}, h, \vec{y}(t_0), \vec{y}(t_1), \ldots, \vec{y}(t_k)) = O(h^{p+1})$$

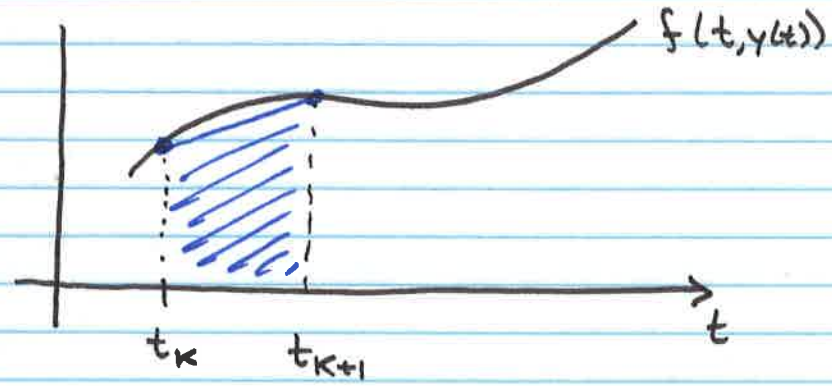
recall our calculations with e_k and δ_k ...

if we set $y(t_k) = y_k$ the "new" error is just the local error.

Other Methods

Recall our integral version of the derivation of Euler's Method.

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(t,y)\,dt$$



A better approximation of the area under the curve between t_k and t_{k+1} ... is the trapezoid method

$$y(t_{k+1}) \approx y(t_k) + \frac{1}{2}(t_{k+1}-t_k)\left[f(t_k, y(t_k)) + f(t_{k+1}, y(t_{k+1}))\right]$$

So the <u>Trapezoid Method</u> is... (for $\vec{y}' = \vec{f}(t, \vec{y})$)

$$\boxed{\vec{y}_{k+1} = \vec{y}_k + \frac{1}{2}h\left[\vec{f}(t_k, \vec{y}_k) + \vec{f}(t_{k+1}, \vec{y}_{k+1})\right]}$$

Observations:

- This is not an explicit method (implicit)

MORE ON THIS LATER!

Q: what order is this method?

Q: Is the trapezoid rule convergent?

Order of Trapezoid Method (characterizing the local error)

start with

$$\vec{y}_{k+1} = \vec{y}_k + \tfrac{1}{2}h\left[\vec{f}(t_k,\vec{y}_k) + \vec{f}(t_k,\vec{y}_{k+1})\right]$$

+ plug in exact solution (use $\vec{y}' = f(t,\vec{y})$)

$$Q \equiv \vec{y}(t_{k+1}) - \left\{\vec{y}(t_k) + \tfrac{1}{2}h\left[\vec{y}'(t_k) + \vec{y}'(t_{k+1})\right]\right\}$$

$$Q = \cancel{\vec{y}(t_k)} + \cancel{\vec{y}'(t_k)}h + \tfrac{1}{2}h^2\cancel{y''}(t_k) + \tfrac{1}{3!}h^3y'''(t_k) + O(h^4)$$

$$- \left\{\cancel{\vec{y}(t_k)} + \tfrac{1}{2}h\left[\cancel{\vec{y}'(t_k)} + \left(\cancel{\vec{y}'(t_k)} + h\cancel{\vec{y}''}(t_k) + \tfrac{1}{2}h^2y'''(t_k)\right)\right]\right\}$$

$Q = O(h^3)$ i.e. local error looks like $O(h^3)$.

so $\boxed{\text{Trapezoid Method is of order 2}}$

We suspect that the global error is $O(h^2)$ and that the Trapezoid Method converges but let's prove this result...

$$\boxed{\text{Thm 1.2} \quad \text{The Trapezoid rule is convergent}}$$

To show this..., let $\vec{e}_{k+1} = \vec{y}(t_{k+1}) - \vec{y}_{k+1}$, $\vec{e}_k = \vec{y}(t_k) - \vec{y}_k$

Taylor $\quad \vec{y}(t_{k+1}) \approx \vec{y}(t_k) + h\vec{y}'(t_k) + \frac{1}{2}h^2\vec{y}''(t_k) + \frac{1}{3!}h^3\vec{y}'''(t_k) + O(h^4)$

Trap $\quad \vec{y}_{k+1} = \vec{y}_k + \frac{1}{2}h\left[\vec{f}(t_k, \vec{y}_k) + \vec{f}(t_{k+1}, \vec{y}_{k+1})\right]$

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subtract:

$$y(t_{k+1}) - y_{k+1} = \vec{y}(t_k) - \vec{y}_k$$

$\vdots$

We could use this approach if we assume f is differentiable so that we can deal with the y'' term.

However, lets use the result we just found, namely

$$y(t_{k+1}) = y(t_k) + \frac{1}{2}h\left[f(t_k, y(t_k)) + f(t_{k+1}, y(t_{k+1}))\right] + O(h^3).$$

so

$$\vec{y}(t_{k+1}) = \vec{y}(t_k) + \tfrac{1}{2}h\left[\vec{f}(t_k,\vec{y}(t_k)) + \vec{f}(t_{k+1},\vec{y}(t_{k+1}))\right] + O(h^3)$$

$$\vec{y}_{k+1} = \vec{y}_k + \tfrac{1}{2}h\left[\vec{f}(t_k,\vec{y}_k) + \vec{f}(t_{k+1},\vec{y}_{k+1})\right]$$

$$\vec{y}(t_{k+1}) - \vec{y}_{k+1} = \vec{y}(t_k) - \vec{y}_k + \tfrac{1}{2}h\Big[\big(\vec{f}(t_k,\vec{y}(t_k)) - \vec{f}(t_k, y(t_k) - \vec{e}_k)\big)$$
$$+ \big(\vec{f}(t_{k+1},\vec{y}(t_{k+1})) - \vec{f}(t_{k+1},\vec{y}(t_{k+1}) - \vec{e}_{k+1})\big)\Big]$$
$$+ O(h^3)$$

$$\|\vec{e}_{k+1}\| = \Big\|\vec{e}_k + \tfrac{1}{2}h\big[(\qquad) + (\qquad)\big] + O(h^3)\Big\|$$

triangle inequality

$$\leq \|\vec{e}_k\| + \tfrac{1}{2}h\|\vec{f}(t_k,\vec{y}(t_k)) - \vec{f}(t_k,\vec{y}(t_k) - \vec{e}_k)\|$$
$$+ \tfrac{1}{2}h\|\vec{f}(t_{k+1},\vec{y}(t_{k+1})) - \vec{f}(t_{k+1},\vec{y}(t_{k+1}) - \vec{e}_{k+1})\| + ch^3$$

$$\leq \|\vec{e}_k\| + \tfrac{1}{2}h\lambda\|\vec{e}_k\| + \tfrac{1}{2}h\lambda\|\vec{e}_{k+1}\| + ch^3$$

so

$$(1 - \tfrac{1}{2}h\lambda)\|\vec{e}_{k+1}\| \leq (1 + \tfrac{1}{2}h\lambda)\|\vec{e}_k\| + ch^3$$

(assuming $1 - \tfrac{1}{2}h\lambda > 0$)

$$\boxed{\|\vec{e}_{k+1}\| \leq \frac{1 + \frac{1}{2}h\lambda}{1 - \frac{1}{2}h\lambda}\|\vec{e}_k\| + \frac{ch^3}{(1 - \frac{1}{2}h\lambda)}}$$

(compare with Euler $\|e_{k+1}\| \leq (1 + h\lambda)\|e_k\| + ch^2$)

Again this has the form

$$e_{k+1} \leq a\, e_k + b$$

$$e_1 \leq b$$

$$e_2 \leq ab + b = (a+1)b$$

$$e_3 \leq a(a+1)b + b = (a^2+a+1)b$$

$$e_k \leq \underbrace{(a^{k-1} + \dots + a + 1)}_{\frac{1-a^k}{1-a}} b$$

so

$$\|\vec{e}_k\| \leq \frac{ch^3}{(1-\frac{1}{2}h\lambda)} \left[\frac{1 - \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^k}{1 - \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)} \right] = \frac{ch^3 \left[1 - \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^k \right]}{(\cancel{1} - \frac{1}{2}h\lambda) - (\cancel{1} + \frac{1}{2}h\lambda)}$$

$$\boxed{\|\vec{e}_k\| \leq \underbrace{\frac{ch^2}{\lambda} \left[\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda} \right)^k - 1 \right]}}$$

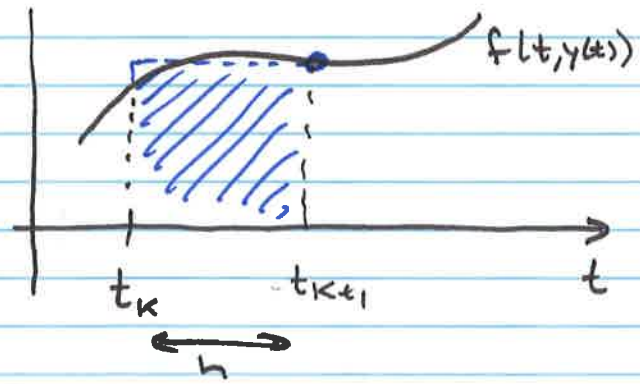
one can argue ... see book (Iserles, p. 9+10) that this term is $O(h^2)$ as $h \to 0$.

$$\boxed{\text{Therefore, trapezoid method converges and its associated global error is } O(h^2)}$$ *

one order better than Euler!

Other Methods

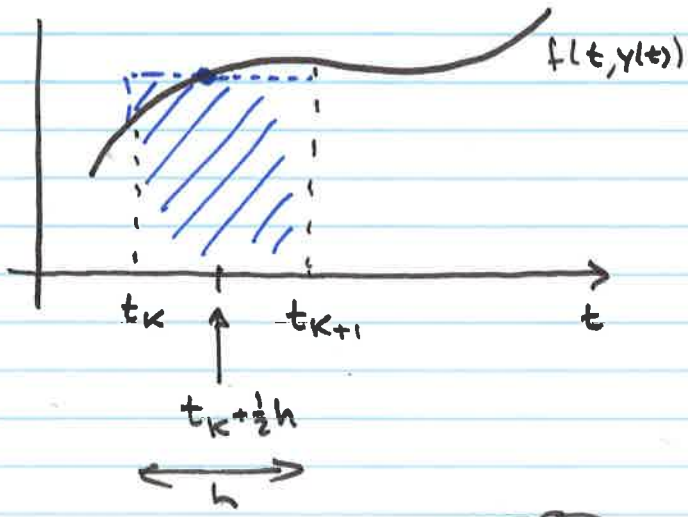
$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(t,y)\,dt$$



approximate based on right endpoint

$$\Rightarrow \boxed{y_{k+1} = y_k + h\,f(t_{k+1}, y_{k+1})}$$

- This is <u>Backward Euler</u> — Implicit
- First order accurate
- Converges



Midpoint rule

$$y_{k+1} = y_k + h\,f\Big(t_k + \tfrac{1}{2}h,\ \underbrace{\tfrac{1}{2}(y_k + y_{k+1})}_{\text{used for } y(t_k+\frac{1}{2}h)}\Big)$$

- <u>Midpoint Rule</u> - Implicit
- Second Order Accurate
- Converges

see HW Exercise 1.1

Heun's Method (Modified Euler)

This method is based on the trapezoid method but is modified to make it ~~an~~ an explicit method

Trap: $$y_{k+1} = y_k + \tfrac{1}{2}h\left[f(t_k, y_k) + f(t_{k+1}, y_{k+1})\right]$$

Heun:

$$\boxed{\begin{aligned} &y_{k+1} = y_k + \tfrac{1}{2}h\left[K_1 + K_2\right] \\ &\text{where} \\ &\quad K_1 = f(t_k, y_k) \\ &\quad K_2 = f(t_{k+1}, \underbrace{y_k + hf(t_k, y_k)}_{\text{effectively an Euler approximation of } y_{k+1}}) \end{aligned}}$$

Comments on Heun's Method

- This is an explicit method
- Verify order of accuracy
- Verify convergence

$\Big\}$ 2nd order

Theta Method

$$\boxed{y_{k+1} = y_k + h\left[\theta f(t_k, y_k) + (1-\theta) f(t_{k+1}, y_{k+1})\right]}$$

Comments:

$\theta = 1$: Euler (explicit)

$\theta = \frac{1}{2}$: Trapezoid (implicit)

$\theta = 0$: Backward Euler (implicit)

$\theta =$ other ... provides flexibility ... extra tuning parameter.

The text shows that (see p.14)

$$Q = y(t_{k+1}) - y(t_k) - h\left[\theta f(t_k, y(t_k)) + (1-\theta) f(t_{k+1}, y(t_{k+1}))\right]$$

$$= \underbrace{\left(\theta - \tfrac{1}{2}\right) h^2 y''(t_k) + \left(\tfrac{1}{2}\theta - \tfrac{1}{3}\right) h^3 y'''(t_k) + O(h^4)}$$

in principle the best term to knock out of the local error is the $O(h^2)$ term. However, in practice there may be situations where knocking out the $O(h^3)$ term makes the actual error lower ($\theta = 2/3$).