Math 113: MATHEMATICA Assignment 2

Due: Thursday, November 21, 4:30pm

Instructions: You may work on this in groups of up to three people. If you work in a group please hand in *only one* copy with all of the group members' names on it. An *environmental fee* will be charged to groups that do not follow this rule. NOTE: My definition of a group includes equal contributions from all members on all problems. All group members should be able to explain any result turned in for credit on this assignment. If your group falls apart mid-way through the assignment please state clearly who did which problem. ALSO NOTE: With probability very close to one, no two groups' assignments should look alike. If four of you work together and print off two versions of the same assignment pretending to be two groups I will know and will reduce scores by an appropriate factor (2 in this example).

IMPORTANT: Answer questions in a neat and organized manner. Specifically, this means that you should use complete sentences and make sure that any graph you expect me to understand (and grade!) has axes and curves labelled (you can do this by hand after you print the graph if that is easiest). Note that different colored curves look the same when printed in black and white. Also, the quantity of pages, graphs and numerical output does not necessarily correspond to quality. Do not turn in Mathematica output without accompanying explanations. Feel free to suppress failed attempts when printing pages.

You have two weeks to complete this assignment. Plan to finish it early. There will be 10% deduction in your score if you do not turn in a hard copy of this assignment at the beginning of class on the due date or earlier. Every subsequent calendar day (counting weekend days and spring break days) late will result in an additional deduction (two days = 20%, three days = 30%, etc.) until I have completed grading the assignments at which time no further late assignments will be accepted.

Pro Tip: *There are helpful commands listed at the end of this assignment. Read through them and make sure you can successfully run them before starting on the assignment.*

Pro Tip 2: *There are multiple formats in which you can enter commands in Mathematica. I recommend (and describe in these notes) the 'Wolfram Languange input [default]'. If you choose to use another format such as 'Free-form input' or 'Wolfram : Alpha query' the recommendations and examples I've listed may not be very helpful.*

1. (*25 pts*) Consider the function

$$f(x) \quad = \quad \ln\left[\frac{\sqrt{x^2+1}}{(3+\sin x)^2}\right]$$

on the x interval $[-10, 10]$.

(a) Define this function in Mathematica and then have Mathematica compute its derivative. Show that this derivative result agrees with the result you obtain by hand, calculating the derivative using logarithmic differentiation. Show all of your work.

Pro Tip: *Mathematica might be quicker than your hand calculations but each will be a valuable validation of the other. If they don't agree, find out why.*

(b) Plot $f(x)$ for $x \in [-10, 10]$.

(c) Plot $f'(x)$ for $x \in [-10, 10]$.

Pro Tip: *Use Mathematica to compute the expression for $f'(x)$ and then plot that.*

(d) Plot $f''(x)$ for $x \in [-10, 10]$.

Pro Tip: *Use Mathematica to compute the expression for $f''(x)$ and then plot that.*

(e) Estimate the critical numbers of $f(x)$. Just focus on the ones on the interval $(-10, 10)$.

(f) Use the First Derivative Test with the help of the graph you've made in part (c) to identify the locations (values of x) where f has local minima and maxima. Do so without direct reference to the graph of f in part (b). Don't worry about endpoints -10, 10.

Pro Tip: *Pretend you don't know what the graph of f looks like on $[-10, 10]$, how do you figure out the local min/max values of f?*

(g) Use the Second Derivative Test with the help of the graphs you've made in parts (c) and (d) to identify the locations (values of x) where f has local minima and maxima. Do so without direct reference to the graph of f in part (b). Don't worry about endpoints -10, 10.

Pro Tip: *Your answers should serve as a check to your answers in part (f).*

2. (*25 points*) We've learned that a linear approximation to a function $g(x)$, near a point $x = a$, is given by

$$\ell_1(x) = g(a) + g'(a)(x - a) \tag{1}$$

Improved approximations can be obtained by using higher derivatives and these approximations have the form

$$\begin{aligned} g(x) \approx \ell_N(x) &= g(a) + g'(a)(x-a) + \frac{1}{2!}g''(a)(x-a)^2 \\ &\quad + \frac{1}{3!}g'''(a)(x-a)^3 + \ldots + \frac{1}{N!}g^{(N)}(a)(x-a)^N \end{aligned} \tag{2}$$

where $g^{(N)}(a)$ is the N^{th} derivative of g at $x = a$ and $N! = N(N-1)(N-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. In Calculus II you'll encounter these approximations in the context of Taylor Series and Taylor Expansions.

(a) For $g(x) = \sin x$ and $a = 0$ define the approximations

$$\begin{aligned}
\ell_1(x) &= g(0) + g'(0)x \\
\ell_3(x) &= g(0) + g'(0)x + \frac{1}{3!}g'''(0)x^3 \\
\ell_5(x) &= g(0) + g'(0)x + \frac{1}{3!}g'''(0)x^3 + \frac{1}{5!}g^{(5)}(0)x^5 \\
\ell_7(x) &= g(0) + g'(0)x + \frac{1}{3!}g'''(0)x^3 + \frac{1}{5!}g^{(5)}(0)x^5 + \frac{1}{7!}g^{(7)}(0)x^7
\end{aligned}$$

Comment: note even derivatives of $g(x) = \sin x$ evaluated at zero are zero and so we only need to consider the odd cases.

Pro Tip: *To define a function like $\ell_5(x)$ in Mathematica once you have properly defined the function $g(x)$ use*

```
gline5[x_] := g[0] + g'[0] * x + (1/6) * g'''[0] * x^3 + (1/120) * g'''''
[0] * x^5
```

Create four plots. The first has $g(x)$ and $\ell_1(x)$. The second has $g(x)$ and $\ell_3(x)$. The third has $g(x)$ and $\ell_5(x)$. The fourth has $g(x)$ and $\ell_7(x)$. Comment on the approximation of the original function $g(x) = \sin x$ in each case. Pick the x and y intervals so that you can see easily the comparison between the function and the approximation.

(b) For the function in problem 1, that is,

$$f(x) = \ln\left[\frac{\sqrt{x^2+1}}{(3+\sin x)^2}\right]$$

and the interval $x \in [-1, 1]$ (instead of $[-10, 10]$) create 5 plots and comment on the approximations. Plot i should show $f(x)$ and $\ell_i(x)$ where

$$\begin{aligned}
\ell_1(x) &= f(0) + f'(0)x \\
\ell_2(x) &= f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 \\
\ell_3(x) &= f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 \\
\ell_4(x) &= f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \frac{1}{4!}f^{(4)}(0)x^4 \\
\ell_5(x) &= f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \frac{1}{4!}f^{(4)}(0)x^4 + \frac{1}{5!}f^{(5)}(0)x^5
\end{aligned}$$

Pro Tip: *Don't compute these derivatives by hand, have Mathematica do the work. Some of these calculations and plots may take a few minutes for Mathematica to execute the command – be patient!*

Some helpful Mathematica commands (in the default Mathematica input format):

• π is `Pi`, e^x is `Exp[x]`, $\ln x$ is `Log[x]`, $\log_b x$ is `Log[b,x]`, $\sin x$ is `Sin[x]`, $\cos x$ is `Cos[x]`, $\tan x$ is `Tan[x]`, $|x|$ is `Abs[x]`, $\tan^{-1} x$ is `ArcTan[x]`. For example, note that if you enter `Sin x , sin x , sin(x), sinx, Sinx, Sin(x)` instead of `Sin[x]` don't expect Mathematica to know what you are talking about.

• Define a function $g(x) = x^2 + 4x + 3$:

```
g[x_] := x^2 + 4*x + 3
```

Note the underscore ('blank') after the x.

• The derivative of a function $g(x)$ can be computed in a number of ways:

`g'[x]` OR `D[g[x],x]`

• It may be helpful to define a new function that is the derivative $g'(x)$:

```
gp[x_]:=g'[x]
```

• The second derivative of $g(x)$ can be computed as:

`g''[x]` OR `D[g[x],{x,2}]` OR `gpp[x_]:=gp'[x]`

where the function `gp` is defined as above. As far as I can tell the 'prime' notation continues to work for higher derivatives (e.g. `g''''''''''[x]` is the 10^{th} derivative of $g(x)$ but `D[g[x],{x,10}]` also works).

• Evaluate $g(0.01)$ as a floating point number try

`N[g[0.01]]` OR `NumberForm[N[g[0.01]], 12]` for 12 decimal places listed

• To define a function of more than one variable (e.g. $G_1(x, h) = x + h$):

```
G1[x_,h_] := x + h
```

• Display a graph (for a properly defined function):

```
Plot[g[x], {x, xmin, xmax}, PlotRange ->{ymin, ymax}]
OR
Plot[g[x], {x, xmin, xmax}]
OR
Plot[G1[x,0.5], {x, xmin, xmax}]
```