Quiz 10 - Friday, April 26

Lecture 4.8 - Antiderivatives

Post-Survey (II) - also Friday, April 26 in recitation - extra credit!

Last time: Integration (Ch. 5)

One application:

**Definition of Definite Integral**

\[
\lim_{N \to \infty} \left( \sum_{i=1}^{N} f(x_i^*) \Delta x \right)
\]

- $\Delta x$ is the width
- $f(x_i^*)$ is the height

The Definite Integral of $f(x)$ on $[a, b]$.

Also

\[
\int_{a}^{b} f(x) \, dx = A_1 - A_2 + A_3
\]

$A_1 > 0$

$A_2 > 0$

$A_3 > 0$
Recall early definition of derivative (limit definition)

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

**Example**

\[ f(x) = x^2 - x \]

Then by definition

\[ f'(x) = \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} \]

\[ = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} \]

\[ = \lim_{h \to 0} \frac{2xh + h^2 + x}{h} = \lim_{h \to 0} 2x + h + 1 \]

\[ = 2x + 1 = f'(x) \]

Let's try to compute the value of a definite integral by using our definition

\[ \int_{a}^{b} f(x) \, dx = \lim_{N \to \infty} \left( \sum_{i=1}^{N} f(x_i^*) \Delta x \right) \]

\( x_i^* \) sample point
Example

Use the definition of definite integral to evaluate

\[ \int_0^1 x \, dx \]

Note:

- \( f(x) = x \)
- Since the region is triangular, its area is \( \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2} \)
- So we expect \( \int_0^1 x \, dx = \frac{1}{2} \).

To use the limit definition - Riemann Sums

\[
\int_a^b f(x) \, dx = \lim_{N \to \infty} \left( \sum_{i=1}^{N} f(x_i^*) \Delta x \right)
\]

Here: \( f(x) = x \), \( a = 0 \), \( b = 1 \)

\[ \Delta x = \frac{1}{N} \]

\[ x_i = \frac{i}{N} \]

\[ i = 0, 1, \ldots, N \]

\[ x_i^* = \text{Sample point on interval } i \]

Choose right endpoint of each slice.

\[ x_i^* = x_i \]
\[ f(x) = x \]
\[ f(x_i^*) = x_i^* = x_i = \frac{i}{N} \]
\[ \Delta x = \frac{1}{N} \]

So
\[
\int_0^1 x \, dx = \lim_{N \to \infty} \left( \frac{N}{N} \sum_{i=1}^{N} \frac{i}{N} \cdot \frac{1}{N} \right)
\]
\[
= \lim_{N \to \infty} \left( \sum_{i=1}^{N} \frac{i}{N^2} \right) - \lim_{N \to \infty} \left( \frac{1}{N^2} + \frac{2}{N^2} + \ldots + \frac{N}{N^2} \right)
\]
\[
= \lim_{N \to \infty} \left( \frac{1}{N^2} \sum_{i=1}^{N} i \right)
\]
\[
\sum_{i=1}^{N} i = \frac{N(N+1)}{2}
\]
\[
= \lim_{N \to \infty} \left( \frac{1}{N^2} \frac{N(N+1)}{2} \right)
\]
\[
= \lim_{N \to \infty} \frac{N^2 + N}{2N^2}
\]
\[
= \lim_{N \to \infty} \frac{1 + \frac{1}{N}}{2} = \frac{1}{2}
\]

Agrees with our triangle area argument.

So
\[
\int_0^1 x \, dx = \frac{1}{2}
\]

In principle, definite integrals can be evaluated using the limit definition of Riemann sums.
Some other integral examples...

\[ \int_{-1}^{1} x \, dx \]
\[ = \frac{1}{2} - \frac{1}{2} = 0 \]

Symmetry/Geometric Argument

\[ \int_{-1}^{1} |x| \, dx \]
\[ = \frac{1}{2} + \frac{1}{2} = 1 \]

\[ \int_{\pi}^{0} \sin x \, dx \]
\[ = A_1 - A_1 = 0 \]

\[ \int_{-4}^{4} \sqrt{16-x^2} \, dx \]
\[ = \frac{1}{2} \pi (4)^2 = 8 \pi \]

Geometry + Area of Half Circle,
\[ y = \sqrt{16-x^2} \]
\[ y^2 + x^2 = 16 = c^2 \]
General Properties of Definite Integrals

- \( \int_{a}^{a} f(x) \, dx = 0 \)

- \( \int_{a}^{b} f(x) \, dx = - \int_{b}^{a} f(x) \, dx \)

- \( \int_{a}^{b} c \, dx = c \cdot (b-a) \)

- \( \int_{a}^{b} [f(x) + g(x)] \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \)

- \( \int_{a}^{b} c \cdot f(x) \, dx = c \cdot \int_{a}^{b} f(x) \, dx \)
\[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \]

**Example**

\[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \]

\[ = c_1(c - a) + c_2(b - c) \]

If \( m \leq f(x) \leq M \) on \([a, b]\),

then \( m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a) \)
5.4 Fundamental Theorem of Calculus

Think about
\[ \int_{a}^{x} f(t) \, dt \]
\( t \) is called an integration variable.

\[ \int_{a}^{x} f(t) \, dt = \int_{a}^{x} f(s) \, ds = \int_{a}^{x} f(p) \, dp = \ldots \]

EX

Let \( A(x) = \int_{0}^{x} t \, dt = \frac{1}{2} x^2 \)

\( f(t) = t \)

Area = \( \frac{1}{2} x \cdot x = \frac{1}{2} x^2 \)

So \( A(x) = \frac{1}{2} x^2 \) (area function)

\[ \frac{dA}{dx} = A'(x) = \frac{d}{dx} \int_{0}^{x} f(t) \, dt \]

Note, for this example (at least)

\( A(x) = \int_{0}^{x} f(t) \, dt \)

Coincidence or not?
Make this more general ... (general $f(t)$)

Let $A(x) = \int_{a}^{x} f(t) \, dt$ “Area function”

We are wondering about $A'(x)$ so let’s use the limit definition of the derivative:

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

$$A(x+h) = \int_{a}^{x+h} f(t) \, dt$$

$$A(x) = \int_{a}^{x} f(t) \, dt$$

$$A(x+h) - A(x) = \int_{x}^{x+h} f(t) \, dt$$

So

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \approx \lim_{h \to 0} \frac{f(x) \cdot h}{h} = f(x)$$

It appears that when

$$A(x) = \int_{a}^{x} f(t) \, dt$$

then

$$A'(x) = f(x)$$