Quiz 9 - Friday, April 19

4.5 L'Hopital's Rule
4.6 Optimization Problems

Today - one more 4.6 Example

4.7 Newton's Method
4.8 Antiderivatives --- Integration

One more optimization example.

EX variation of last example

- Design cylindrical container.
- Holds a fixed volume $V_0$
- Top + bottom material cost $D$ $/ per unit area
- Side material costs $1$ $/ per unit area.

Find the least expensive container that holds volume $V_0$.

Some geometry: $V_0 = \pi r^2 h$ constraint

$\frac{h}{\pi r^2}$
Objective Function: Cost of container.

\[ C = C_{\text{top}} + C_{\text{bot}} + C_{\text{side}} \]

\[ = D \cdot \pi r^2 + D \cdot \pi r^2 + 1 \cdot (2 \pi r \cdot h) \]

\[ C = 2D\pi r^2 + 2\pi rh \quad \therefore h = \frac{V_0}{\pi r^2} \]

\[ C = 2D\pi r^2 + 2\pi r \left( \frac{V_0}{\pi r^2} \right) \]

\[ C = 2D\pi r^2 + \frac{2V_0}{r} \]

\[ \text{Plan: Choose } r \text{ to minimize this cost.} \]

To find minimum of \( C \)

- Find critical points of \( C \)

\[ \frac{dc}{dr} : C' = 2D\pi \cdot 2r - 2V_0 \cdot r^{-2} \]

\[ = 4\pi DR - \frac{2V_0}{r^2} = \frac{1}{r^2} \left( 4\pi DR^3 - 2V_0 \right) \]

\[ = \frac{4\pi D}{r^2} \left( r^3 - \frac{2V_0}{4\pi D} \right) \]

\[ C'(r) = \frac{4\pi D}{r^2} \left( r^3 - \frac{V_0}{2\pi D} \right) \]

\[ \frac{dc}{dr} = 0 \quad r^3 = \frac{V_0}{2\pi D} \]

\[ \text{critical number} \quad \Rightarrow r = \left( \frac{V_0}{2\pi D} \right)^{1/3} \]
So the first Der. Test tells us that

\[
C(r) = \text{cost has a minimum when } \\
\frac{V_0}{(2\pi D)^3} \quad \text{or} \quad r_c = \frac{V_0}{2\pi D}
\]

What about \( h \)?

\[
h = \frac{V_0}{\pi r^2} = \frac{V_0 r}{\pi r^3} = \frac{V_0 r}{\pi (\frac{V_0}{2\pi D})}
\]

\[
h = \frac{r}{(\frac{1}{2D})} = 2Dr
\]

So \( h_c = (2D)r_c \quad r_c = \left(\frac{V_0}{2\pi D}\right)^{\frac{1}{3}} \)

\( D = \text{cost of top + bottom} \)

\( D > 1 \) (expensive top + bottom)

\( D < 1 \) (cheep top + bottom)

\( \Rightarrow \) make height larger than \( D = 1 \) case

\( \Rightarrow \) make height smaller

\( D = 1 \) everything costs the same
Overall Plan

- Draw a diagram - geometry (Volume, Area, Length...)
  - Identify variables

- Identify any constraints

- Identify an objective function
  - What is it that needs to be minimized or maximized?
  - Try to write a formula for this if possible.
  - Can you eliminate a variable? (using the constraint)

- Apply some type of min/max argument
  (e.g. Critical Numbers, First Derivative Test, Second Der. Test...)

- Answer the question. What are the dimensions... what is the value of the objective function.
4.7 Newton's Method

Basic Problem "Solved" by Newton's Method

Find $x$ such that $f(x) = 0$.

\[ f(x) = x^2 - 9 \]

\[
\begin{align*}
f(x) &= 0 \\
(x-3)(x+3) &= 0 \\
x &= 3 \\
x &= -3
\end{align*}
\]

\[ \text{Ex} \quad \text{(Simple case)} \]

\[ f(x) = A x^2 + B x + C \]

\[
\begin{align*}
\text{quadratic formula tells us when} \\
x &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}
\end{align*}
\]

What about a more general function?

\[ f(x) = e^x - 4x \]

Find $x$ s.t. $f(x) = 0$. 
\[ f(x) = (1+x)^2 (1-x)^2 + b \times x \quad b = \text{constant (parameter)} \]

\[ b = 0: \quad f(x) = 0 \quad \text{when} \quad x = -1, 1 \]

**In general**

\[ b < b^* \approx 1.539 \]

3 values of \( x \) where \( f(x) = 0 \).

\[ b > b^* \]

Only one solution.

Don't have a good way to write down exact formulas for these solutions...

Enter Newton's Method.
Newton's Method

Given $f(x)$ where we wish to find $x$ so that $f(x) = 0$.

Initial guess of approximation for $x^*$.  

\[ y = f(x) \]  

Somewhat complicated function.

Find $x_1$.

Evaluate $f(x_1)$ and $f'(x_1)$.

\[ y - f(x_1) = f'(x_1)(x - x_1) \]

\[ 0 - f(x_1) = f'(x_1)(x_2 - x_1) \]

\[ \frac{-f(x_1)}{f'(x_1)} = x_2 - x_1 \]

\[ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \]

New approximation at $x^*$.

Newtton's method uses this over and over.

Repeat $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

In general

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad k = 1, 2, 3, \ldots \]
Newton's method generates a sequence of numbers:

\[ x_1, x_2, x_3, x_4, \ldots \rightarrow x^* \] (hopefully)

- you provide a guess

Newton's method may or may not converge.
- depends on the guess.

**Example**

Apply Newton's method to find a root of

\[ f(x) = x^3 - x + 1 \] (i.e. find \( x \) where \( f(x) = 0 \)).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( f(x_n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-1.5</td>
<td>-0.275</td>
</tr>
<tr>
<td>3</td>
<td>-1.347826</td>
<td>-0.10068</td>
</tr>
<tr>
<td>4</td>
<td>-1.324520</td>
<td>-0.002058</td>
</tr>
<tr>
<td>5</td>
<td>-1.32472</td>
<td>-0.924 \times 10^{-6}</td>
</tr>
<tr>
<td>6</td>
<td>-1.32471</td>
<td>-0.1 \times 10^{-8}</td>
</tr>
</tbody>
</table>

\[ x^* \approx -1.3247 \ldots \]

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]

\[ f(x) = x^3 - x + 1 \]
\[ f'(x) = 3x^2 - 1 \]

\[ x_{k+1} = x_k - \frac{(x_k^3 - x_k + 1)}{(3x_k^2 - 1)} \]

\[ k = 1 \] (iterate)
There is a dependence on good guesses vs. bad guesses!
### 4.8 Antiderivatives

**Definition:**

Let $F(x)$ be a function whose derivative is $f(x)$. That is, $F'(x) = f(x)$. Note also

$$\frac{d}{dx} \left[ F(x) + C \right] = \frac{dF}{dx} + C = F'(x) = f(x)$$

$C$ is a constant.

$F(x)$ is called an antiderivative of $f(x)$.

$F(x) + C$ is the most general antiderivative of $f(x)$.

---

**Some Standard Antiderivatives**

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$F(x)$</th>
<th>Check Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$\frac{1}{2}x^2 + C$</td>
<td>$\frac{d}{dx} \left( \frac{1}{2}x^2 + C \right) = x + 0^+$</td>
</tr>
<tr>
<td>$x^2$</td>
<td>$\frac{1}{3}x^3 + C$</td>
<td>$\frac{d}{dx} \left( \frac{1}{3}x^3 + C \right) = x^2 + 0^+$</td>
</tr>
<tr>
<td>$A$ (constant)</td>
<td>$Ax + C$</td>
<td>$\frac{d}{dx} (Ax + C) = A$</td>
</tr>
</tbody>
</table>

**Additional Antiderivatives:**

<table>
<thead>
<tr>
<th>$x^n$ ($n \neq -1$)</th>
<th>$\frac{1}{n+1}x^{n+1} + C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^{-1}$</td>
<td>$\ln</td>
</tr>
</tbody>
</table>
\[ f(x) \]

\[ F(x) \]

\[
\sin(ax) \\
\text{a = constant}
\]

\[
-\frac{\cos(ax)}{a} + C
\]

\[
\int \left[ -\frac{1}{a} \cos(ax) + C \right] \\
= \frac{1}{a} \sin(ax) + C
\]

\[ \checkmark \]

\[ F'(x) = f(x) \]

\[ \cos(ax) \\
\text{a = constant} \]

\[
\frac{1}{a} \sin(ax) + C
\]

\[ \checkmark \]