Quiz 2: Friday, Feb. 8
recitation

2.1 - idea of limits
2.2 - limit laws
2.3 - precise def. (definition + concept)
2.4 - one-sided limits
2.5 - continuity

Exam 1: Mon., Feb. 18
Ch. 1, Ch. 2
Part of Ch. 3

Last time: the limit laws (2.2)
the precise definition of limit (2.3)

The precise definition made less precise...

\[
\lim_{x \to a} f(x) = L
\]

means

if you need to make \( f(x) - L \) \( \leq \epsilon \) for \( \forall \epsilon > 0 \),
you can do it (by making \( 0 < |x-a| < \delta \)).

2.5 Continuity (Continuous Functions)

Def: A function is continuous at \( x = a \) if

\[
\lim_{x \to a} f(x) = f(a)
\]

(1) Left = right limits match
(2) The function must be defined at \( x = a \), \( f(a) \)

\[
\begin{align*}
\text{Continuous at } x = a & \quad \text{Not continuous at } x = a \\
\end{align*}
\]
Definition: A function is continuous on an interval if it is continuous at every point on the interval.

- $f$ is continuous on $(b,c)$
- $f$ is continuous on $(d,e)$
- $f$ is not continuous on $(c,d)$
- $f$ is not continuous on $(b,e)$

What general functions are continuous?

1. Polynomials are continuous for all real numbers.
2. Rational functions are continuous at all points in their domain.

Example: $f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$, $D$: all reals, except $x=0$.

- $f$ is continuous on all intervals that do not include $x=0$.
- $\lim_{x \to 0} f(x) = \pm \infty$
3. Roots e.g. \( f(x) = \sqrt{x} \)

are continuous on all points in their domain.

4. Trig Functions are continuous on their domains.

\( f(x) = \sin x \)

\( f(x) = \cos x \)

\( f(x) = \tan x = \frac{\sin x}{\cos x} \)
5. Exponential and Logarithmic Functions are continuous on their domains.

What about continuity of piecewise-defined functions?
- Beware of transition points

**Example**

\[ f(x) = \begin{cases} 
  x + 1 & \text{if } x \leq 0 \\
  \sin x & \text{if } x > 0 
\end{cases} \]

(transition point is \( x = 0 \))

This function \( f(x) \) is not continuous at \( x = 0 \).
EX

Given

\[ f(x) = \begin{cases} 
A \cos x & x < 0 \\
B & x = 0 \\
\sqrt{x - 1} & 0 < x < 1 \\
\frac{x}{x-1} & x = 1 \\
C & x > 1 \\
\end{cases} \]

A, B, C, D are constants.
Transition points are x = 0, x = 1.

Q: Can we choose constants A, B, C, D so that this function is continuous for all values of x?

For continuity at x = 0, set

we require

\[ \lim_\limits{x \to 0^+} f(x) = \lim_\limits{x \to 0^-} f(x) = f(0) \]

\[ \lim_\limits{x \to 0^+} f(x) = \lim_\limits{x \to 0^-} \sqrt{x-1} = \frac{-1}{2} = [1] \]

\[ \lim_\limits{x \to 0^-} f(x) = \lim_\limits{x \to 0^-} A \cos x = A \cdot 1 = [A] \]

f(0) = [B]

Choose A = 1, B = 1
For continuity at \( x = 1 \), we require
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)
\]

- \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{D}{x} = D \)
- \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{\sqrt{x} - 1}{(x - 1)(\sqrt{x} + 1)} \cdot \lim_{x \to 1^-} \frac{1}{\sqrt{x} + 1} = \frac{1}{2} \)

\( f(1) = C \)

Choose \( C = \frac{1}{2}, D > \frac{1}{2} \)

Sketch...

\( y = f(x) \) with \( A = B = 1, \ C = D = \frac{1}{2} \)
Composite Functions \( f(g(x)) = f \circ g(x) \)

Composite Limit Theorem

If \( \lim_{x \to a} g(x) = L \) and if \( f \) is continuous at \( L \) (i.e., \( \lim_{x \to L} f(x) = f(L) \)) then

\[
\lim_{x \to a} f(g(x)) = f\left( \lim_{x \to a} g(x) \right) = f(L)
\]

Intermediate Value Theorem (IVT)

- Observe that \( f \) is continuous on \([a, b]\).
- Think about a horizontal line \( y = N \) with \( f(a) < N < f(b) \).

**IVT:** Suppose \( f \) is a continuous function on a closed interval \([a, b]\) and \( N \) is any number between \( f(a) \) and \( f(b) \). Then there exists at least one number \( c \) on \([a, b]\) such that \( f(c) = N \).
Note that the IVT does not apply to a function that is not continuous.

\[ f(x) = \sqrt{x} - 1.5 \]

Use the IVT to show that \( f \) has at least one root (a value of \( x \) where \( f(x) = 0 \)) on the closed interval \([1, 4]\).

**Note:** \( f(x) \) is continuous on \([1, 4]\).

- \( f(1) = \sqrt{1} - 1.5 = 1 - 1.5 = -0.5 < 0 \).
- \( f(4) = \sqrt{4} - 1.5 = 2 - 1.5 = 0.5 > 0 \).

The IVT says \( f \) must be zero at least once between 1 and 4.

\[ \sqrt{c} - 1.5 = 0. \]
More generally, if \( f(a) < 0 \) and \( f(b) > 0 \) with \( f \) continuous on \([a,b]\), then by the IVT, there is at least one value \( x = c \) where \( f(c) = 0 \).

Next up is 2.6 Asymptotes, Limits at Infinity, and Infinite Limits.

**Formal Definition of Limit at Infinity**

**Def:**

We say \( \lim_{x \to \infty} f(x) = L \) as \( x \) approaches infinity \( x \) is \( L \) if for every \( \varepsilon > 0 \) there exists a corresponding value \( M \) such that for all \( x \) in the domain of \( f \) we have:

\[ |f(x) - L| < \varepsilon \text{ whenever } x > M. \]
$$\exists \epsilon \quad f(x) = \frac{1}{x}$$

$$\lim_{x \to 0} \frac{1}{x} = 0$$

$$\phi(x) = \frac{1}{x}$$

$$\epsilon \quad \epsilon \quad \epsilon$$

$$x \to 0$$