Ch. 1 Functions of various sorts

\[ y = f(x) \]

- Domain of \( f \)
- Range of \( f \)
Types of Functions

- Linear Functions: \( f(x) = mx + b \)
  \( m, b \) constants

- Power Functions: \( f(x) = x^r \) \( r \) constant
  
  e.g. \( f(x) = x^2 \)
  \( f(x) = x^3 \)
  \( f(x) = x^{-1/2} = \sqrt{x} \)
  
  Domain: \([0, \infty)\)
  Range: \([0, \infty)\)

- Polynomials

- Rational Functions: ratios of polynomials

- Exponential + Logarithmic Functions

- Trig Functions: \( \sin x, \cos x, \tan x \)

- Inverse Trig Functions

Combinations + Modifications of these functions:

**Example**

\( f(x) = e^x \)

\( D: (-\infty, \infty) \)
\( R: (0, \infty) \)

\( g(x) = 4e^{x-8} \)

\( g(0) = 4e^{-8} \)
shifting/stretching...

\[ f(x) = \sqrt{x} \]

\[ D: \mathbb{R}^+ \quad R: \mathbb{R} \]

\[ f(x) = \frac{x+1}{2} \quad D: \mathbb{R} \quad R: \mathbb{R} \]

\[ f(x) = 1 + \frac{1}{100} \sqrt{x-2} \quad D: [2, \infty) \quad R: [1, \infty) \quad f(x) \geq 1 \]

\[ 1 + \frac{1}{100} \sqrt{x-2} \]

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Piecewise-defined functions

\[ f(x) = \begin{cases} 
-3 & \text{if } x \leq 0 \\
\frac{x}{3} & \text{if } 0 < x < 2 \\
x^2 & \text{if } x \geq 2 
\end{cases} \]

\[ f(-5) = -3 \quad f(1.7) = 1.7 \quad f(6) = 36 \]
Composite Function

Given two functions \( f \) and \( g \), the composite function, denoted by \( f \circ g \), is defined by:

\[
(f \circ g)(x) = f(g(x))
\]

**Example**

\[
\begin{align*}
\text{Let } f(x) &= x^3 + 3, \\
g(x) &= x^2
\end{align*}
\]

\[
\begin{align*}
g(2) &= 4, \\
(f \circ g)(2) &= f(g(2)) = f(4) = 4^3 + 3 = 64 + 3 = 67, \\
g(f(x)) &= g(x^3 + 3) = (x^3 + 3)^2, \\
&= x^6 + 6x^3 + 9
\end{align*}
\]
\[ f(g(x)) = f(x^2) = (x^2)^3 + 3 = x^6 + 3 \]

Notice \( f(g(x)) \neq g(f(x)) \) in general.

Other variations
- composite functions with 3 or more functions...

\[ f(g(h(x))) \]
\[ = f(g(\cos x)) \]
\[ = f((\cos x)^2) \]
\[ = (\cos^2 x)^3 + 3 \]
\[ = (\cos x)^6 + 3 = \cos^6 x + 3 \]

In mathematics
\[ B = (\cos [x])^6 + 3 \]
Inverse Functions

Definition: Given a function $f$, the inverse of $f$ (if it exists) is a function $g$ (usually denoted by $f^{-1}$) such that whenever

$$y = f(x) \text{ then } g(y) = x$$

(i.e. $f^{-1}(y) = x$)

When does such an inverse function exist?

- The function $f$ must be one-to-one. That is, a one-to-one function $f(x)$ has $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in the domain of $f$.

This function has no inverse.
- Passes horizontal line test

This function does not have an inverse.
- Fails horizontal line test
Note: connection to composite functions.

For a function $f$ with an inverse $f^{-1}$

\[ f^{-1}(f(x)) = x \quad \text{for } x \text{ in the domain of } f \]
\[ f(f^{-1}(x)) = x \quad \text{for } x \text{ in the domain of } f^{-1} \]

**Ex.**

\[ y = f(x) = x^3 + 1 \]

<table>
<thead>
<tr>
<th>$f(0)$</th>
<th>$f^{-1}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Graphs of $f(x)$ and $f^{-1}(x)$ on the same graph.
Exponential and logarithmic functions are examples of inverse functions

More generally,

\[ f(x) = b^x \quad \quad f^{-1}(x) = \log_b x \]

\( b = \text{any positive } \neq 1 \) 

That is,

\[ y = b^x \quad \iff \quad \log_b y = x \]

Ex: Find \( x \) if \( 16 = 2^x \), \( x = \log_2 16 = 4 \)
Inverse Trig Functions

**EX**

Inverse sine "arcsin"

Consider

\[ y = f(x) = \sin x \]

**D:** \((-\infty, \infty)\)

**R:** \([-1, 1]\)

To define the inverse sine function we need to restrict the domain (since \(f(x) = \sin x\) is not one-to-one).

Note \(\sin(\frac{\pi}{2}) = 1\) but also \(\sin\left(-\frac{3\pi}{2}\right) = 1\)

\(\sin\left(\frac{\pi}{2} + n \cdot 2\pi\right) = 1\)

With this restriction we define

\[ y = f^{-1}(x) = \sin^{-1} x = \arcsin x \]

such that \(\sin y = x\). That is, \(y = \arcsin x\)

\[ \sin y = x \]
Note: The domain of $f'(x) = \arcsin x$ is $[-1, 1]$

the range of $f^{-1}(x) = \arcsin x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$