Midterm Exam #2—Math 101, Section 205

March 13, 2015

Duration: 50 minutes

Surname (Last Name) ____________ Given Name ____________ Student Number ____________

Do not open this test until instructed to do so! This exam should have 9 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work. Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked. Continue on the back of the page if you run out of space.

UBC rules governing examinations:

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
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   (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (c) purposely viewing the written papers of other examination candidates;
   (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem	Out of	Score	Problem	Out of	Score
1	6		5	8	
2	6		6	8	
3	6		7	8	
4	3		**Total**	45	

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

1a. **[3 pts]** Write out the general form for the partial fractions expansion of

$$\frac{x - x^3 + 7}{(x^3 + 10x^2 + 26x)(x-3)^3}.$$

Do not solve for the constants!

We can factor $x^3 + 10x^2 + 26x$ as $x \cdot (x^2 + 10x + 26)$. The quadratic $x^2 + 10x + 26$ has negative discriminant, hence no real roots, so cannot be factored further. Notice that the degree of the numerator is less than the degree of the denominator so we do not need to perform long division. Finally, noticed that the factor $(x-3)$ appears 3 times. Therefore, the partial fractions decomposition is of the form:

$$\frac{A}{x} + \frac{Bx + C}{x^2 + 10x + 26} + \frac{D}{x-3} + \frac{E}{(x-3)^2} + \frac{F}{(x-3)^3}.$$

One point is awarded for the correct contribution above corresponding to each factor.

1b. **[3 pts]** Find the limit of the following sequence:

$$e^{\sqrt{3}}, e^{\sqrt{\sqrt{3}}}, e^{\sqrt{\sqrt{\sqrt{3}}}}, \ldots, e^{\sqrt[2^n]{3}}, \ldots$$

Simplify your answer.

Notice that the sequence $\left\{\frac{1}{2^n}\right\}_n$ converges to 0. The function $f(x) = 3^x$ is continuous, hence the sequence $\left\{3^{\frac{1}{2^n}}\right\}_n$ converges to $3^0 = 1$. The function $\exp(x) = e^x$ is also continuous, hence our original sequence $\left\{e^{\sqrt[2^n]{3}}\right\}_n$ converges to $e^1 = e$. For this solution, one point is awarded for the correct expression $\left\{3^{\frac{1}{2^n}}\right\}_n$, one point is awarded for remarking $\frac{1}{2^n} \to 0$, and the final point is reserved for the correct answer. Alternatively, one can simply notice that the function $g(x) = e^{3^{\frac{1}{2^x}}}$ is continuous for $x > 0$ and converges to e when $x \to \infty$, and our sequence is simply $\{g(n)\}_n$.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

2a. **[3 pts]** A straight rod of negligible thickness and of length l meters has one end at the origin and is parallel to the x-axis. Suppose that the density of the portion of the rod x meters from the origin is $\rho(x) = \sqrt[3]{x}$ kilograms per meter. How far from the origin is the center of mass in terms of l?

The mass of the rod is given by integrating the density:

$$m = \int_0^l \sqrt[3]{x}\,\mathrm{d}x = \frac{3}{4}x^{\frac{4}{3}}|_0^l = \frac{3}{4}l^{\frac{4}{3}} \text{ kg.}$$

One point is awarded for the correct mass. The center of mass is then given by

$$\begin{aligned}\overline{x} = \frac{1}{m}\int_0^l x\rho(x)\,\mathrm{d}x &= \frac{1}{m}\int_0^l x\sqrt[3]{x}\,\mathrm{d}x \\ &= \frac{4}{3}l^{-\frac{4}{3}} \cdot \left[\frac{3}{7}x^{\frac{7}{3}}|_0^l\right] \\ &= \frac{4}{7}l.\end{aligned}$$

One point is awarded for the correct expression for $\overline{x}$ and the final point is awarded for the correct answer.

2b. **[3 pts]** **[3 pts]** Find the average value of the function

$$f(t) = \frac{\sqrt{t^2 - 1}}{t}$$

on the interval $\left[1, \sqrt{2}\right]$. *Simplify your answer.*

By the Mean Value Theorem for Integrals, the average value of the function is given by

$$\frac{1}{\sqrt{2}-1}\int_1^{\sqrt{2}} f(t)\,\mathrm{d}t = \frac{1}{\sqrt{2}-1}\int_1^{\sqrt{2}} \frac{\sqrt{t^2-1}}{t}\,\mathrm{d}t.$$

This observation (if made correctly) is worth one point. Now let $t = \sec\theta$. Then $dt = \sec\theta\tan\theta d\theta$, and since this is really the substitution $\theta = \sec^{-1} t$, $\sqrt{t^2-1} = \tan\theta$. Hence we have the average value is

$$\begin{aligned}
\frac{1}{\sqrt{2}-1}\int_1^{\sqrt{2}} \frac{\sqrt{t^2-1}}{t}\,\mathrm{d}t &= \frac{1}{\sqrt{2}-1}\int_{\sec^{-1}1}^{\sec^{-1}\sqrt{2}} \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta d\theta\,\mathrm{d}\theta \\
&= \frac{1}{\sqrt{2}-1}\int_0^{\frac{\pi}{4}} \tan^2\theta\,\mathrm{d}\theta \\
&= \frac{1}{\sqrt{2}-1}\int_0^{\frac{\pi}{4}} \left(\sec^2\theta - 1\right), \mathrm{d}\theta \\
&= \frac{1}{\sqrt{2}-1}\left(\tan\theta - \theta|_0^{\frac{\pi}{4}}\right) \\
&= \frac{1}{\sqrt{2}-1}\left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right) \\
&= \frac{1-\frac{\pi}{4}}{\sqrt{2}-1}.
\end{aligned}$$

One point is awarded for getting as far as

$$\frac{1}{\sqrt{2}-1}\int_0^{\frac{\pi}{4}} \tan^2\theta\,\mathrm{d}\theta$$

with the correct end-points. The final point is reserved for the correct answer with accompanying work.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

3a. **[3 pts]** Calculate the following indefinite integral:

$$\int \sin^5(\theta)\cos^2(\theta)\ \mathrm{d}\theta.$$

$$\begin{aligned}\int \sin^5(\theta)\cos^2(\theta)\ \mathrm{d}\theta &= \int \left(\sin^2(\theta)\right)^2\cos^2(\theta)\ \sin(\theta)\,\mathrm{d}\theta \\ &= \int \left(1-\cos^2(\theta)\right)^2\cos^2(\theta)\ \sin(\theta)\,\mathrm{d}\theta\end{aligned}$$

One point is awarded for getting this far. Now let $u = \cos(\theta)$. Then we get

$$\begin{aligned}\int \left(1-\cos^2(\theta)\right)^2\cos^2(\theta)\ \sin(\theta)\,\mathrm{d}\theta &= -\int \left(1-u^2\right)^2 u^2\,\mathrm{d}u \\ &= -\int \left(u^6-2u^4+u^2\right)\ \mathrm{d}u \\ &= -\frac{u^7}{7}+2\frac{u^5}{5}-\frac{u^3}{3}+C \\ &= -\frac{\cos^7(\theta)}{7}+\frac{2}{5}\cos^5(\theta)-\frac{\cos^3(\theta)}{3}+C\end{aligned}$$

One point is awarded for reducing to $-\int\left(1-u^2\right)^2 u^2\,\mathrm{d}u$. The final point is reserved for the correct answer together with the accompanying work.

3b. **[3 pts]** Calculate the following definite integral:

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2-10x+26}$$

By definition we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2-10x+26} = \lim_{s\to-\infty}\int_s^0 \frac{\mathrm{d}x}{x^2-10x+26} + \lim_{t\to\infty}\int_0^t \frac{\mathrm{d}x}{x^2-10x+26}.$$

One point for correctly splitting the integral as the sum of the limit of two integrals as above. Notice that

$$\int_0^t \frac{\mathrm{d}x}{x^2-10x+26} = \int_0^t \frac{\mathrm{d}x}{(x-5)^2+1}$$

(one point for completing the square) and letting $u = x-5$, we have

$$\begin{aligned} \int_0^t \frac{\mathrm{d}x}{(x-5)^2+1} &= \int_{-5}^{t-5} \frac{\mathrm{d}u}{u^2+1} \\ &= \tan^{-1}(t-5) - \tan^{-1}(-5). \end{aligned}$$

Similarly, $\int_s^0 \frac{\mathrm{d}x}{x^2-10x+26} = \tan^{-1}(-5) - \tan^{-1}(s-5)$. Hence

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2-10x+26} &= \lim_{s\to-\infty} -\tan^{-1}(s-5) + \lim_{t\to\infty}\tan^{-1}(t-5) \\ &= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi. \end{aligned}$$

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

4. **[3 pts]** Recall that the error E_{S_n} in using Simpson's rule with n-rectangles of equal length to calculate

$$\int_a^b f(x)\, \mathrm{d}x$$

is bounded by

$$|E_{S_n}| \le \frac{K(b-a)^5}{180n^4},$$

where K is a number such that $|f^{(4)}(x)| \le K$ on $[a,b]$. Calculate the **exact** error involved in using Simpson's rule for calculating

$$\int_0^7 \left(t^3 - 19t^2 + 11\right)\, \mathrm{d}t$$

when using 14 rectangles. *Simplify your answer.*

The fourth derivative of $t^3 - 19t^2 + 11$ is identically equal to zero, hence we can take $K = 0$ above and conclude that for any n the absolute value of the error in using Simpson's rule with n-rectangles is zero, i.e. it gives the exact answer! One point for deducing that we can take $K = 0$, another point for deducing $|E_{S_n}| \le 0$ and the final point for realizing that this implies $|E_{S_n}| = 0$.

Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

5.

(a) **[6 pts]** Find the general solution to the differential equation

$$y' = \pi e^x \left(1 + y^2\right) \cos(3x)$$

We can solve the differential equation by separation of variables:

$$\begin{aligned}
\frac{dy}{dx} &= \pi e^x \left(1+y^2\right)\cos(3x) \\
\frac{dy}{1+y^2} &= \pi e^x \cos(3x)\, dx \\
\int \frac{dy}{1+y^2} &= \pi \int e^x \cos(3x)\, dx \\
\tan^{-1}(y) &= \pi \int e^x \cos(3x)\, dx.
\end{aligned}$$

One point for seperating the equation by bringing x and y to different sides, and another point for realizing that one should then integrate both sides. A third point for correctly integrating with respect to y. To calculate the integral with respect to x, we can use integration by parts with $u = e^x$ and $dv = \cos x dx$. Then

$$\int e^x \cos(3x)\, dx = e^x \frac{\sin 3x}{3} - \frac{1}{3} \int e^x \sin(3x)\, dx.$$

One point for performing integration by parts correctly. We can calculate the resulting integral again by integration by parts, with $u = e^x$ and $dv = \sin(3x)\, dx$, and we get

$$\int e^x \sin(3x)\, dx = -e^x \frac{\cos(3x)}{3} + \frac{1}{3} \int e^x \cos(3x)\, dx.$$

Putting this all together we get

$$\int e^x \cos(3x)\, dx = e^x \frac{\sin 3x}{3} + \frac{1}{9} e^x \cos(3x) - \frac{1}{9} \int e^x \cos(3x)\, dx.$$

Hence we have

$$\int e^x \cos(3x)\, dx = \frac{3e^x \sin(3x) + e^x \cos(3x)}{10} + C,$$

(one point for this) and the general solution is therefore

$$y(x) = \tan\left(\pi \frac{3e^x \sin(3x) + e^x \cos(3x)}{10} + C\right)$$

(final point for this correct answer)

(b) **[2 pts]** Find the unique solution to part a) for which $y(0) = 1$.

Using the above general solution, we see that $y(0) = \tan\left(\frac{\pi}{10} + C\right)$. For this to be equal to 1 we need $\frac{\pi}{10} + C = \frac{\pi}{4} + n \cdot \pi$ for some n. However, since $\tan(x + n\pi) = \tan(x)$ for all x, the extra factors of π don't matter, so we get $C = \frac{3}{20}\pi$ (one point). Hence the solution we seek is

$$y(x) = \tan\left(\pi \frac{3e^x \sin(3x) + e^x \cos(3x)}{10} + \frac{3}{20}\pi\right).$$

(one point)

6. **[8 pts]** Calculate the following indefinite integral:

$$\int \frac{x^7 + 2x^6 - 15x^5 - 30x - 30}{x^3 + 2x^2 - 15x}\,\mathrm{d}x$$

We start by putting the rational integrand in proper form. $x^7 + 2x^6 - 15x^5 - 30x - 30 = (x^3 + 2x^2 - 15x)\,x^4 - 30x - 30$ so

$$\frac{x^7 + 2x^6 - 15x^5 - 30x - 30}{x^3 + 2x^2 - 15x} = x^4 - \frac{30x + 30}{x^3 + 2x^2 - 15x}$$

(but this can also be directly seen by long division)- one point for getting this far (if no long division is performed, the maximum score for this problem is 4, and only if there are no other errors). In order to expand $\frac{30x+30}{x^3+2x^2-15x}$ in terms of partial fractions, we first factor the denominator:

$$x^3 + 2x^2 - 15x = x\,(x+5)\,(x-3)\,.$$

(one point for this factorization). So, the expansion will be of the form

$$\frac{30x + 30}{x^3 + 2x^2 - 15x} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-3}.$$

(one point for the correct form of the partial fractions expansion). By expanding the right hand side and comparing numerators, we must have that for all x,

$$A\,(x+5)\,(x-3) + Bx\,(x-3) + Cx\,(x+5) = 30x + 30.$$

(one point for this equation) Setting $x = 0$ yields $30 = -15A$ so $A = -2$. Setting $x = -5$ yields $40B = -120$ so $B = -3$. Setting $x = 3$ yields $24C = 120$ so $C = 5$. Hence

$$\frac{x^7 + 2x^6 - 15x^5 - 30x - 30}{x^3 + 2x^2 - 15x} = x^4 + \frac{2}{x} + \frac{3}{x+5} - \frac{5}{x-3},$$

(one point) and therefore

$$\int \frac{x^7 + 2x^6 - 15x^5 - 30x - 30}{x^3 + 2x^2 - 15x}\,\mathrm{d}x = \frac{x^5}{5} + 2\ln|x| + 3\ln|x+5| - 5\ln|x-3| + C$$

(final 2 points).

7. The goal of this exercise is to integrate the function $\sqrt{e^{2x}+1}$.

(a) **[2 pts]** Using substitution, rewrite the integral $\int \sqrt{e^{2x}+1}\,\mathrm{d}x$ as an integral with respect to the variable $u = e^x$.

If $u = e^x$ then $du = e^x dx = udx$ and thus $dx = \frac{du}{u}$. One point for this expression of dx purely in terms of u. Hence we have:

$$\int \sqrt{e^{2x}+1}\,\mathrm{d}x = \int \frac{\sqrt{1+u^2}}{u}\,\mathrm{d}u$$

(one point).

(b) **[2 pts]** Find an appropriate substitution to rewrite the resulting integral from part $a)$ as:

$$\int \frac{w^2}{w^2-1}\,\mathrm{d}w.$$

Show your work. No partial credit will be given for using the incorrect integral from part a).

Let $w = \sqrt{u^2+1}$, (one point for this substitution, one point for the rest as follows:) then $dw = \frac{u}{\sqrt{u^2+1}}du = \frac{u}{w}du$. So $du = \frac{w}{u}dw$, which implies

$$\int \frac{\sqrt{1+u^2}}{u}\,\mathrm{d}u = \int \frac{w^2}{u^2}dw.$$

Note that $u^2 = w^2 - 1$ so finally we have:

$$\int \frac{\sqrt{1+u^2}}{u}\,\mathrm{d}u = \int \frac{w^2}{w^2-1}\,\mathrm{d}w.$$

(c) **[2 pts]** Calculate the indefinite integral

$$\int \frac{w^2}{w^2-1}\,\mathrm{d}w.$$

First notice that

$$\frac{w^2}{w^2-1} = \frac{1}{w^2-1} + \frac{w^2-1}{w^2-1} = 1 + \frac{1}{w^2-1}.$$

We can expand $\frac{1}{w^2-1}$ by partial fractions as

$$\frac{1}{w^2-1} = \frac{A}{w-1} + \frac{B}{w+1} = \frac{(A+B)\,w + (A-B)}{w^2-1},$$

so $A = \frac{1}{2}$ and $B = -\frac{1}{2}$, (one point for getting this far) and hence

$$\int \frac{w^2}{w^2-1}\,\mathrm{d}w = w + \frac{1}{2}\ln|w-1| - \frac{1}{2}\ln|w+1| + C$$

(final point)

(d) **[2 pts]** Calculate the indefinite integral

$$\int \sqrt{e^{2x}+1}\,\mathrm{d}x.$$

We have that $w = \sqrt{u^2+1} = \sqrt{e^{2x}+1}$, (one point) hence by part c):

$$\int \sqrt{e^{2x}+1}\,\mathrm{d}x = \sqrt{e^{2x}+1} + \frac{1}{2}\ln\left(\sqrt{e^{2x}+1}-1\right) - \frac{1}{2}\ln\left(\sqrt{e^{2x}+1}+1\right) + C$$

(one point).