Weekly Homework 5

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Topos Theory

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Problem 1. The underlying space of a sheaf

Let $\mathbb{TOP}$ denote the category of $\mathcal{U}$-small topological spaces and $\widehat{\mathbb{TOP}}$ denote the category of $\mathcal{V}$-small topological spaces, where $\mathcal{U} \in \mathcal{V}$ are both Grothendieck universes, and similarly denote $\mathbf{Set}$ and $\widehat{\mathbf{Set}}$. Let $\mathbf{Sh}(\mathbb{TOP})$ denote the category of sheaves of $\mathcal{U}$-small sets, and $\widehat{\mathbf{Sh}}(\mathbb{TOP})$ the category of sheaves of $\mathcal{V}$-small sets. Let

$$\tau : \mathbb{TOP} \hookrightarrow \widehat{\mathbb{TOP}}$$

be the canonical functor.

(a) Show that the left Kan extension $U := Lan_{\widehat{y}}(\tau) : \widehat{\mathbf{Set}}^{\mathbb{TOP}^{op}} \to \widehat{\mathbb{TOP}}$ of τ along the Yoneda embedding

$$\widehat{y} : \mathbb{TOP} \hookrightarrow \widehat{\mathbf{Set}}^{\mathbb{TOP}^{op}}$$

sends a presheaf F of $\mathcal{V}$-small sets to a space with underlying set $F(*)$, where $*$ denotes the terminal space. Conclude that if F is in the subcategory $\mathbf{Set}^{\mathbb{TOP}^{op}}$, then $U(F)$ is in the subcategory $\mathbb{TOP}$.

(b) Show that the Yoneda embedding $y : \mathbb{TOP} \hookrightarrow \mathbf{Sh}(\mathbb{TOP})$ has a left adjoint.

Problem 2. Compactly Generated Spaces as Sheaves

Recall that a topological space X is **compactly generated** if a subset $A \subseteq X$ is open if $f^{-1}(A)$ is open for all maps $f : T \to X$, with T compact Hausdorff. Denote by $\mathbb{LCH}$ the category of locally compact Hausdorff spaces, and define the functor

$$\varphi : \mathbb{TOP} \to \mathbf{Sh}(\mathbb{LCH})$$

by

$$\varphi(X)(L) = \mathbf{Hom}_{\mathbb{TOP}}(L, X).$$

(a) Show that the essential image of φ in $\mathbf{Sh}(\mathbb{LCH})$ is equivalent to the category of compactly generated spaces.
Hint: Modify arguments from the last exercise to show that φ has a left adjoint.