Weekly Homework 10

Instructor: David Carchedi
Topos Theory

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Problem 1. Epimorphisms and Monomorphisms in a Topos

Let $\mathscr{E}$ be a topos.

(a) Let $m : A \to B$ be a monomorphism in $\mathscr{E}$, and let

$$\phi_m : B \to \Omega$$

be a map to the subobject classifier of $\mathscr{E}$ classifying m, and similarly denote by ϕ_B the map classifying the maximal subobject of B (i.e. id_B). Denote by

$$m' : E = \varprojlim (B \rightrightarrows \Omega) \to B$$

the equalizer diagram for ϕ_m and ϕ_B. Show that m and m' represent the same subobject of B. Deduce that a morphism in a topos is an isomorphism if and only if it is both a monomorphism and an epimorphism.

(b) Let $f : X \to Y$ be a map of *sets*. Denote by

$$X \times_Y X \rightrightarrows X$$

the kernel pair of f and by

$$Y \rightrightarrows Y \coprod_X Y$$

the cokernel pair of f. Show that the coequalizer of the kernel pair and the equalizer of the cokernel pair both coincide with the the set $f(X)$. Deduce that for $f : X \to Y$ a map in $\mathscr{E}$,

$$X \to \varprojlim \left(Y \rightrightarrows Y \coprod_X Y \right) \to Y$$

and

$$X \to \varinjlim (X \times_Y X \rightrightarrows X) \to Y$$

are both factorizations of f by an epimorphism follows by a monomorphism.

(c) Show that the factorization of a morphism f in $\mathscr{E}$ into an epimorphism followed by a monomorphism is unique up to isomorphism. Deduce that $f : X \to Y$ is an epimorphism, if and only if the canonical map

$$\varprojlim (X \times_Y X \rightrightarrows X) \to Y$$

is an isomorphism.

Problem 2. Geometric Morphisms between Presheaf Topoi

Let $\varphi : \mathscr{C} \to \mathscr{D}$ be a functor between small categories. Denote by

$$\varphi^* : \mathbf{Set}^{\mathscr{D}^{op}} \to \mathbf{Set}^{\mathscr{C}^{op}}$$

the obvious restriction functor. Show:

(a) φ^* has a left adjoint $\varphi_! := \mathrm{Lan}_{y_{\mathscr{C}}}\, y_{\mathscr{D}} \circ \varphi$.

(b) φ^* preserves colimits. Deduce that it has a right adjoint φ_* given by

$$\varphi_* (Y) (D) = \mathrm{Hom} (\varphi^* y (D), Y).$$

(c) Show the following are equivalent:

i) The pair (φ_*, φ^*) is a geometric embedding.

ii) The counit $\varphi^* \varphi_* \Rightarrow id$ is an isomorphism.

iii) The unit $id \Rightarrow \varphi^* \varphi_!$ is an isomorphism.

iv) The functor φ is full and faithful.

Problem 3. Étale Geometric Morphisms

Let $k : B \to A$ be a morphism in a topos $\mathscr{E}$. Show that the functor

$$k^* : \mathscr{E}/A \to \mathscr{E}/B$$

induced by pullback has both a left adjoint $\sum_k$ and a right adjoint $\prod_k$. Conclude that the pair $(k_* = \prod_k, k^*)$ constitute a geometric morphism

$$\mathscr{E}/B \to \mathscr{E}/A.$$

Geometric morphisms of this form are called **étale**.