Weekly Homework 1

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Topos Theory

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Problem 1. Actions of Categories

(a) Let $\mathscr{C}$ be a small category. Prove that the category of sets with a left $\mathscr{C}$-action, $\mathscr{C}$-**Set** as defined in lecture, is canonically equivalent (in fact isomorphic) to the category $\mathbf{Set}^{\mathscr{C}}$.

(b) Define for yourself the notion of a right $\mathscr{C}$-action on a set, and show the associated category $\mathbf{Set}-\mathscr{C}$ is canonically equivalent (in fact isomorphic) to the presheaf category $\mathbf{Set}^{\mathscr{C}^{op}}$.

(**Hint:** Spell out what a left $\mathscr{C}^{op}$-action means in terms of $\mathscr{C}$.)

(c) Deduce that if $\mathscr{C}$ is a groupoid, the category sets with a left $\mathscr{C}$-action and the category of sets with a right $\mathscr{C}$-action are isomorphic. (A groupoid is a category in which every arrow is an isomorphism.)

(d) Let $X \circlearrowleft \mathscr{C}$ be a right $\mathscr{C}$-set. Define the **action category** $X \rtimes \mathscr{C}$ to be the category whose objects are the set X, and the arrows are all of the form $(x, f) : x \cdot f \to x$, where x and f are such that f can act on x (via the moment map). Show that this is a well defined category with composition defined by the rule which makes the composition

$$(x \cdot g) \cdot f \xrightarrow{(x \cdot g, f)} x \cdot g \xrightarrow{(x, g)} x$$

equal to $(x, g \circ f)$.

(e) Show that there is a canonical functor $\theta_X : X \rtimes \mathscr{C} \to \mathscr{C}$.

Problem 2. Discrete Fibrations

Definition 1. A functor $F : \mathscr{D} \to \mathscr{C}$ between small categories is said to be a **discrete fibration** if the commutative diagram

$$\begin{array}{ccc} \mathscr{D}_1 & \xrightarrow{F_1} & \mathscr{C}_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ \mathscr{D}_0 & \xrightarrow[F_0]{} & \mathscr{C}_0, \end{array}$$

is a pullback, where the map s associates an arrow its source. Define the category $\mathbf{DFib}(\mathscr{C})$ to be the category whose objects are discrete fibrations over $\mathscr{C}$, and whose arrows are triangles of functors which *strictly commute* over $\mathscr{C}$.

(a) Show that an equivalence of categories is a discrete fibration if and only if it is an isomorphism of categories. Deduce that the isomorphisms in $\mathbf{DFib}(\mathscr{C})$ are isomorphisms in **Cat** over $\mathscr{C}$.

(b) If

$$\begin{array}{ccc} \mathscr{D} & \xrightarrow{H} & \mathscr{E} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & \mathscr{C} & \end{array}$$

is a commutative diagram of functors, with G a discrete fibration, show that F is a discrete fibration if and only if H is. Deduce that if $F : \mathscr{D} \to \mathscr{C}$ is a discrete fibration, then there is a canonical equivalence of categories

$$\mathbf{DFib}(\mathscr{C})/F \simeq \mathbf{DFib}(\mathscr{D}),$$

where $\mathbf{DFib}(\mathscr{C})/F$ is the slice category over F (i.e. the objects are the arrows in $\mathbf{DFib}(\mathscr{C})$ with target F, and the morphisms are given by commutative triangles over F.)

(c) If $X \circlearrowleft \mathscr{C}$ is a $\mathscr{C}$-set, show that the canonical functor $X \rtimes \mathscr{C} \to \mathscr{C}$ from 1. (e) is a discrete fibration. Show that this construction extends to a functor

$$(\,\cdot\,) \rtimes \mathscr{C} : \mathbf{Set} - \mathscr{C} \to \mathbf{DFib}(\mathscr{C}).$$

Describe an inverse (up to natural isomorphism) of this functor, to deduce that $(\,\cdot\,) \rtimes \mathscr{C}$ is an equivalence of categories.

(d) Let F be a presheaf on $\mathscr{C}$. Define the category $\mathscr{C}/F$ to the category whose objects are morphisms $f : y(C) \to F$, with $C \in \mathscr{C}_0$, and whose arrows are given by commutative triangles. There is a canonical functor

$$\pi_F : \mathscr{C}/F \to \mathscr{C}$$

sending $f : y(C) \to F$ to C. Show that this functor is a discrete fibration. Show that this construction canonically extends to a functor

$$\pi : \mathbf{Set}^{\mathscr{C}^{op}} \to \mathbf{DFib}(\mathscr{C}).$$

(e) Denote by

$$\Theta : \mathbf{Set}^{\mathscr{C}^{op}} \to \mathbf{Set} - \mathscr{C}$$

the canonical isomorphism from 1. (b). Show that the diagram

$$\begin{array}{ccc} \mathbf{Set}^{\mathscr{C}^{op}} & \xrightarrow{\quad\Theta\quad} & \mathbf{Set}\text{-}\mathscr{C} \\ & {}_{\pi}\searrow \quad \swarrow_{(\,\cdot\,)\rtimes\mathscr{C}} & \\ & \mathbf{DFib}\,(\mathscr{C}) & \end{array}$$

commutes up to a canonical natural isomorphism.

(f) If X is a presheaf on $\mathscr{C}$, the category $\Theta\,(X) \rtimes \mathscr{C}$ is often denoted as

$$\int\limits_{\mathscr{C}} X,$$

and is called the category of elements of X. Describe this category explicitly in terms of X. Deduce from part (b) and part (e) that there are canonical equivalences

$$\mathbf{Set}^{\mathscr{C}^{op}}/X \simeq \mathbf{Set}^{(\mathscr{C}/X)^{op}} \simeq \mathbf{Set}^{\left(\int_{\mathscr{C}} X\right)^{op}}.$$