A QUICK NOTE ON ÉTALE STACKS

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Abstract. These notes start by closely following a talk I gave at the "Higher Structures Along the Lower Rhine" workshop in Bonn, in January. I then give a taste of some applications to foliation theory. Finally, I give a brief account on the general theory of étale stacks.

1. Étale Differentiable Stacks

Disclaimer: Throughout these notes, manifolds will not be assumed paracompact, 2^{nd}-countable, or even Hausdorff.
Idea: Étale stacks are like manifolds whose points posses intrinsic (discrete) automorphism groups.

Examples:

1) G a discrete group, M a smooth manifold, $G \circlearrowright M$ a smooth action

$$\rightsquigarrow M//G \text{ -"the stacky quotient"}$$

There is a (surjective) projection $\pi : M \to M//G$ making M into a principal G-bundle over $M//G$, and given a point $x \in M$,

$$\mathrm{Aut}\,(\pi\,(x)) \cong G_x \text{ - the stabilizer subgroup.}$$

2) $\mathscr{X}$ locally of the form $M//G_i$, for G_i finite groups $\Leftrightarrow$ $\mathscr{X}$ is an orbifold.
3) $(M, \mathcal{F})$ a foliated manifold,

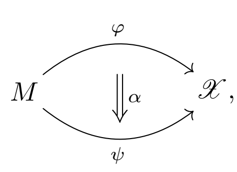
$$M//\mathcal{F} \text{ -the "stacky leaf space."}$$

$L : * \to M//\mathcal{F}$ a leaf has

$$\mathrm{Aut}\,(L) \cong \mathrm{Hol}\,(L) \text{ - holonomy group.}$$

Étale stacks form a bicategory:

Rough Idea: The points of an étale stack $\mathscr{X}$ form not just a set, but a groupoid. If



with M a manifold and $\mathscr{X}$ an étale stack, then for all points $x \in M$,

$$\alpha\,(x) : \varphi\,(x) \to \psi\,(x)$$

is an arrow in this groupoid. In particular, such an arrow must be invertible $\Rightarrow$ $\mathrm{Hom}(M, \mathscr{X})$ is a groupoid.

So given an étale stack $\mathscr{X}$, we get a functor

$$\begin{array}{ccc} \mathrm{Mfd}^{op} & \xrightarrow{\widehat{\mathscr{X}}} & \mathrm{Gpd} \\ M & \longmapsto & \mathrm{Hom}(M, \mathscr{X}). \end{array}$$

This functor completely characterizes $\mathscr{X}$. Instead of describing what an étale stack is geometrically, we can characterize those functors of the form $\widehat{\mathscr{X}}$. This is the "functor of points approach."

What properties must such a functor satisfy?

Need continuity: If $\mathcal{U} = (U_\alpha)$ is an open cover of M, morphisms

$$\varphi_\alpha : U_\alpha \to \mathscr{X}$$

together with coherent isomorphisms

$$\varphi_\alpha|_{U_\alpha \cap U_\beta} \cong \varphi_\beta|_{U_\alpha \cap U_\beta},$$

should be the same as morphisms

$$M \to \mathscr{X},$$

i.e. $\widehat{\mathscr{X}}$ is a *stack.*

To see what other properties we need to characterize $\widehat{\mathscr{X}}$, lets turn back to our examples:

1) There is an action groupoid $G \ltimes M$:

   objects: M
   arrows: $G \times M$, where a pair $(g, x) : x \to g \cdot x$.

   This is a groupoid object in Mfd, and the source and target maps (s, t) are local diffeomorphisms. $\Rightarrow G \ltimes M$ is an étale Lie groupoid, and
   $$M//G \simeq M//\left(G \ltimes M\right)_1.$$

3) There exists a Lie groupoid $\mathrm{Hol}(M, \mathcal{F}) \rightrightarrows M$, whose arrows are given by holonomy classes of leaf-wise paths. $\mathrm{Hol}(M, \mathcal{F})$ is Morita equivalent to an étale Lie groupoid and
   $$M//\mathcal{F} \simeq M//\,\mathrm{Hol}(M, \mathcal{F})_1.$$

3) If $G \circlearrowright M$ with G finite, then $G \ltimes M$ is étale and proper (Meaning $(s, t) : G \times M \to M \times M$ is proper) and this is a local property. Conversely, if $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ is étale and proper, $\mathcal{G}$ is locally of the form $G_i \ltimes M_i$. So orbifolds are stacky quotients of the form $\mathcal{G}_0//\mathcal{G}_1$ for étale and proper Lie groupoids.

Idea: Étale stacks are "quotients" of the form $\mathcal{G}_0//\mathcal{G}_1$ for an étale Lie groupoid $\mathcal{G}$.

More precisely:

$$\begin{array}{rcl} \mathcal{G} \text{ is a Lie groupoid } \rightsquigarrow \tilde{y}(\mathcal{G}) : \mathrm{Mfd}^{op} & \to & \mathrm{Gpd} \\ M & \mapsto & \mathrm{Hom}(M, \mathcal{G}_1) \rightrightarrows \mathrm{Hom}(M, \mathcal{G}_0) . \end{array}$$

In general $\tilde{y}(\mathcal{G})$ is not a stack. But the inclusion

$$\mathrm{St}(\mathrm{Mfd}) \hookrightarrow \mathrm{Gpd}^{\mathrm{Mfd}^{op}}$$

of stacks into general presheaves of groupoids has left-adjoint a which canonically associates to a given presheaf of groupoids its *stackification.*

Definition. Let $\mathcal{G}_0//\mathcal{G}_1 := a(\tilde{y}(\mathcal{G}))$. Stacks of this form are called differentiable stacks.

A smooth functor $\mathcal{G} \to \mathcal{H}$ is a Morita equivalence $\Leftrightarrow \mathcal{G}_0//G_1 \to \mathcal{H}_0//\mathcal{H}_1$ an equivalence. $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent if and only if there is a diagram of Morita equivalences

$$\begin{array}{ccc} & \mathcal{K} & \\ \swarrow & & \searrow \\ \mathcal{G} & & \mathcal{H}. \end{array}$$

Maps $\mathcal{G}_0//\mathcal{G}_1 \to \mathscr{L}_0//\mathscr{L}_1$ are the same as diagrams

$$\begin{array}{ccc} & \mathcal{K} & \\ \swarrow & & \searrow \\ \mathcal{G} & & \mathscr{L} \end{array}$$

with $\mathcal{K} \to \mathcal{G}$ a Morita equivalence. They may also be described by principal $\mathscr{L}$ bundles over $\mathcal{G}$ (Hilsum-Skandalis maps.)

Definition. $\mathscr{X} \in \mathrm{St}(\mathrm{Mfd})$ is an étale stack $\Leftrightarrow \mathscr{X} \simeq \mathcal{G}_0//\mathcal{G}_1$, with $\mathcal{G}$ Morita equivalent to an étale Lie groupoid.

1.1. **Sheaf Theory.** *Recall:* If M is a manifold, and $F \in \mathrm{Sh}(M)$ a sheaf, there exists an étalé space

$$\underline{L(F)} \xrightarrow{L(F)} M,$$

which is a local diffeomorphism such that sections of $L(F)$ over an open subset $U \subset M \overset{1:1}{\longleftrightarrow}$ elements of $F(U)$

$$\left(\underline{L(F)} = \coprod_{x \in M} F_x, \right)$$

and we have an adjoint equivalence

$$\mathrm{Sh}(M) \underset{L}{\overset{\Gamma}{\rightleftarrows}} Et/M \,,$$

with $L \dashv \Gamma$, where Et/M is the category of local diffeomorphisms over M.
We will generalize this for étale stacks:

Definition. A morphism $\mathcal{G}_0//\mathcal{G}_1 \to \mathcal{H}_0//\mathcal{H}_1$ with $\mathcal{G}$ and $\mathcal{H}$ étale is a local diffeomorphism if and only if it corresponds to a diagram

$$\begin{array}{ccc} & \mathcal{K} & \\ {}^{Mor.\sim}\swarrow & & \searrow{}^{\varphi} \\ \mathcal{G} & & \mathcal{H} \end{array}$$

with φ_0 a local diffeomorphism.

Given an étale Lie groupoid, the canonical projection $\mathcal{G}_0 \to \mathcal{G}_0//\mathcal{G}_1$ is always a local diffeomorphism.

Define a category $\mathrm{Site}(\mathcal{G})$:

objects $\mathcal{O}(\mathcal{G}_0)$ open subsets.
arrows

$$\begin{array}{ccccc} U & & \xrightarrow{\quad f \quad} & & V \\ & \searrow & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \swarrow & \\ & \mathcal{G}_0 & & \mathcal{G}_0 & \\ & & \searrow \quad \swarrow & & \\ & & \mathcal{G}_0//\mathcal{G}_1 & & \end{array} .$$

$\mathrm{Sh}(\mathrm{Site}(\mathcal{G})) \simeq \mathcal{B}\mathcal{G}$ - the category of equivariant sheaves on $\mathcal{G}_0$. (The classifying topos of $\mathcal{G}$.) If $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent, $\mathcal{B}\mathcal{G} \simeq \mathcal{B}\mathcal{H}$, so if $\mathscr{X} \simeq \mathcal{G}_0//\mathcal{G}_1$, one may define its category of sheaves to be sheaves over $\mathrm{Site}(\mathcal{G})$, and this is well defined. Even more, one may define the 2-category of stacks of groupoids over $\mathscr{X}$, by

$$\mathrm{St}(\mathscr{X}) \simeq \mathrm{St}(\mathrm{Site}(\mathcal{G})).$$

Theorem (D.C). *For an étale differentiable stack $\mathscr{X} \simeq \mathcal{G}_0//\mathcal{G}_1$, there is an adjoint equivalence*

$$\mathrm{St}(\mathscr{X}) \underset{L}{\overset{\Gamma}{\rightleftarrows}} Et/\mathscr{X},$$

where $Et/\mathscr{X}$ is the bicategory of local diffeomorphisms $\mathscr{Y} \to \mathscr{X}$, with $\mathscr{Y}$ an étale differentiable stack. Moreover, for each $\mathscr{Z} \in \mathrm{St}(\mathscr{X})$, and $U \subset \mathcal{G}_0$ open, $\mathscr{Z}(U)$ is equivalent to the groupoid of sections

$$\begin{array}{ccccc} & & & & \underline{L(\mathscr{Z})} \\ & & {}^{\sigma}\nearrow & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle L(\mathscr{Z})} \\ U & \hookrightarrow & \mathcal{G}_0 & \longrightarrow & \mathscr{X}. \end{array}$$

$L(\mathscr{Z})$ is called the étalé realization of $\mathscr{Z}$. Notice that even when $\mathscr{X} = M$ is a manifold, this gives something new, as it says any stack of groupoids on a manifold comes from sections of a local diffeomorphism $\mathscr{Y} \to M$ from an étale differentiable stack.

1.2. **Haefliger Groupoids and Effectivity.**

Definition. Let M be a manifold, $H(M)$ is the étale Lie groupoid, called the Haefliger groupoid of M, whose objects are M and whose arrows are germs of locally defined diffeomorphisms.

Given $\mathcal{G}$ étale and $x \xrightarrow{g} y$ in $\mathcal{G}$, choose a neighborhood $U \ni g$ over which s and t are injective

$$\begin{aligned} &\rightsquigarrow \quad t \circ (s|_U)^{-1} \text{ is a local diffeomorphism } x \mapsto y \\ &\rightsquigarrow \quad \mathcal{G} \to H(\mathcal{G}_0) \text{ (which is a local diffeomorphism).} \end{aligned}$$

Definition. $\mathcal{G}$ is <u>effective</u> $\Leftrightarrow \mathcal{G} \to H(\mathcal{G}_0)$ is faithful.

In fact, we can construct an étale Lie groupoid out of these germs to produce $\mathrm{Eff}(\mathcal{G})$, the <u>effective part</u> of $\mathcal{G}$, and $\mathcal{G}$ is effective $\Leftrightarrow \mathcal{G} \to \mathrm{Eff}(\mathcal{G})$ is an isomorphism. Effectivity is Morita invariant, therefore we can make the following definition:

Definition. An étale differentiable stack $\mathscr{X}$ is <u>effective</u> if $\mathscr{X} \simeq \mathcal{G}_0//\mathcal{G}_1$ for $\mathcal{G}$ an effective étale Lie groupoid.

Geometric meaning: For an étale stack $\mathscr{X} \simeq \mathcal{G}_0//\mathcal{G}_1$, with $\pi : \mathcal{G}_0 \to \mathscr{X}$ the canonical quotient map, and a point $x \in \mathcal{G}_0$, the automorphism group $\mathrm{Aut}(\pi(x))$ acts on the germ of $\mathcal{G}_0$ around x. $\mathscr{X}$ is effective if and only if each of these actions are faithful.

Proposition. *For M and N connected manifolds,*

$$H(M) \simeq_{Mor.} H(N) \Leftrightarrow \dim(M) = \dim(N).$$

Definition. Let

$$\mathbb{H} := \coprod_{n=0}^{\infty} \left(\mathbb{R}^n // H(\mathbb{R}^n)_1\right) \simeq \left(\coprod_{n=0}^{\infty} \mathbb{R}^n\right) // H\left(\coprod_{n=0}^{\infty} \mathbb{R}^n\right)_1.$$

$\mathbb{H}$ is the <u>universal étale stack.</u>

Given an étale stack $\mathscr{X} \simeq \mathcal{G}_0//\mathcal{G}_1$, with $\dim(\mathcal{G}_0) = n$, we have the composite

$$\mathscr{X} \simeq \mathcal{G}_0//\mathcal{G}_1 \to \mathcal{G}_0//H(\mathcal{G}_0)_1 \simeq \mathbb{R}^n//H(\mathbb{R}^n)_1 \hookrightarrow \mathbb{H},$$

denoted by $ef_{\mathscr{X}}$. It is a local diffeomorphism, hence corresponds to a stack of groupoids $Ef_{\mathscr{X}}$ over $\mathbb{H}$.

Theorem (D.C). *$\mathscr{X}$ is effective $\Leftrightarrow Ef_{\mathscr{X}}$ is in fact a sheaf (of sets) over $\mathbb{H}$.*

Theorem (D.C). *$\mathbb{H}$ is a terminal object in the bicategory $\mathcal{E}t\mathcal{S}t^{et}$ of étale differentiable stacks and local diffeomorphisms, and hence*

$$\mathcal{E}t\mathcal{S}t^{et} \simeq \mathcal{E}t\mathcal{S}t^{et}/\mathbb{H} \simeq Et/\mathbb{H} \simeq \mathrm{St}(\mathbb{H}).$$

Theorem (D.C). *Let Mfd^{et} denote the category of smooth manifolds and local diffeomorphisms. Then there is a canonical equivalence*

$$\omega : \mathrm{St}(\mathbb{H}) \xrightarrow{\sim} \mathrm{St}\left(\mathrm{Mfd}^{et}\right).$$

Moreover, the induced equivalence

$$\mathcal{E}t\mathcal{S}t^{et} \simeq \mathrm{St}\left(\mathrm{Mfd}^{et}\right)$$

sends a manifold M to its representable sheaf.

What this last theorem basically says is, if you have any stack $\mathscr{Z} \in \mathrm{St}\left(\mathrm{Mfd}^{et}\right)$, that is some *moduli problem* which is functorial with respect to local diffeomorphisms, then there exists a unique étale differentiable stack $\tilde{\mathscr{Z}}$, such that for a given manifold M, the groupoid $\mathscr{Z}(M)$ is equivalent to the groupoid of local diffeomorphisms from M to $\tilde{\mathscr{Z}}$. Conversely, given any étale differentiable stack $\mathscr{X}$, it determines a stack on manifolds and local diffeomorphisms by assigning a manifold M the groupoid of local diffeomorphisms from M to $\mathscr{X}$. These operations are inverse to each other.

As an example: Let $R : \left(\mathrm{Mfd}^{et}\right)^{op} \to \mathrm{Set}$ be the functor which assigns a manifold M its set of Riemannian metrics. This is not even a functor on Mfd, but it is functorial with respect to local diffeomorphisms, and in fact is a sheaf. So there exists an étale differentiable stack $\mathcal{R}$, such that local diffeomorphisms

$$M \to \mathcal{R}$$

are the same as Riemannian metrics on M. We call $\mathcal{R}$ the *classifying stack for Riemannian metrics* (In fact, the proof of the equivalence tells you not only the existence of such an $\mathcal{R}$, but also an explicit étale Lie groupoid model.)

Corollary. *There is a canonical equivalence* $\mathcal{E}t\mathcal{S}t^{et}_{\mathit{eff}} \simeq \mathrm{Sh}\left(\mathrm{Mfd}^{et}\right)$, *between effective étale differentiable stacks and local diffeomorphisms, and sheaves on* Mfd^{et}.

Given any stack $\mathscr{Z}$ of groupoids, one can always take isomorphism classes objectwise to get a presheaf of sets, and then sheafify the result to get a sheaf, denoted by $\pi_0(\mathscr{Z})$. This produces a left adjoint π_0 to the inclusion of sheaves (of sets) into stacks (of groupoids).

Theorem (D.C). *Under the equivalences*

$$\mathcal{E}t\mathcal{S}t^{et} \simeq \mathrm{St}\left(\mathrm{Mfd}^{et}\right)$$

and

$$\mathcal{E}t\mathcal{S}t^{et}_{\mathit{eff}} \simeq \mathrm{Sh}\left(\mathrm{Mfd}^{et}\right),$$

the left adjoint $\pi_0 : \mathrm{St}\left(\mathrm{Mfd}^{et}\right) \to \mathrm{Sh}\left(\mathrm{Mfd}\right)^{et}$ *corresponds to the functor sending an étale stack* $\mathscr{X}$ *to its effective part* $\mathrm{Eff}(\mathscr{X})$.

Let $j : \mathrm{Mfd}^{et} \to \mathrm{Mfd}$ be the canonical inclusion. Given any stack $\mathscr{Y} \in \mathrm{St}(\mathrm{Mfd})$, one may restrict it to Mfd^{et} to get a stack $j^*\mathscr{Z}$, hence producing a *restriction functor*

$$j^* : \mathrm{St}(\mathrm{Mfd}) \to \mathrm{St}\left(\mathrm{Mfd}^{et}\right).$$

This functor actually has a left adjoint,

$$j_! : \mathrm{St}\left(\mathrm{Mfd}^{et}\right) \to \mathrm{St}(\mathrm{Mfd})$$

called the *prolongation functor.*

Theorem (D.C). *A stack* $\mathscr{X} \in \mathrm{St}(\mathrm{Mfd})$ *is an étale differentiable stack if and only if it is in the essential image of* $j_!$.

1.3. **Applications to Foliation Theory.** Étale differentiable stacks are also enjoy an intimate connection with foliation theory. For example, after recasting into the language of étale stacks, one has the following theorem:

Theorem. *(Haefliger, Moerdijk, Kock) For M a smooth manifold, equivalence classes of submersions*

$$M \to \mathbb{H}$$

are in bijection with regular foliations on M.

(If it factors through $\mathbb{R}^q//H\left(\mathbb{R}^q\right)_1$, then it is a q-codimensional foliation.)

Fact: Given any submersion $\mathscr{X} \to \mathscr{Y}$ between étale stacks, if factors uniquely as

$$\mathscr{X} \to \mathscr{Y}' \to \mathscr{Y}$$

with $\mathscr{X} \to \mathscr{Y}'$ a submersion with connected fibers, and $\mathscr{Y}' \to \mathscr{Y}$ and local diffeomorphism which encodes a sheaf. (Not all local diffeomorphisms encode a sheaf, as the sections could form a groupoid rather than a set.) Moreover, if $\mathcal{F} : M \to \mathbb{H}$ is a submersion classifying a foliation on M, the factorization is given by

$$M \to M// \operatorname{Hol}\left(M, \mathcal{F}\right)_1 \to \mathbb{H},$$

where $\operatorname{Hol}\left(M, \mathcal{F}\right)$ is the holonomy groupoid of the foliation, and the map

$$M// \operatorname{Hol}\left(M, \mathcal{F}\right)_1 \to \mathbb{H}$$

is the unique local diffeomorphism (since $\mathbb{H}$ is terminal). We denote

$$M//\mathcal{F} := M// \operatorname{Hol}\left(M, \mathcal{F}\right)$$

and call it the stacky leaf space. When the leaf space happens to be a manifold, it agrees.

Example 1. Let $\mathcal{R}$ be the classifying stack for Riemannian metrics. Suppose that

$$M \to \mathcal{R}$$

is a *submersion*. Then it factors uniquely as $M \to \mathcal{R}' \to \mathcal{R}$, with $M \to \mathcal{R}'$ a submersion with connected fibers, and $\mathcal{R}' \to \mathcal{R}$ a sheaf. However, one can also consider the composite

$$M \to \mathcal{R} \to \mathbb{H}$$

by the unique local diffeomorphism $\mathcal{R} \to \mathbb{H}$. This is a submersion $\mathcal{F} : M \to \mathbb{H}$ so it classifies a foliation. Moreover, it can be factored as the submersion with connected fibers $M \to \mathcal{R}'$ follows by the sheaf $\mathcal{R}' \to \mathcal{R} \to \mathbb{H}$ (this uses that Riemannian metrics form a sheaf not just a stack). By uniqueness, one has that $\mathcal{R}' = M//\mathcal{F}$, so that one gets a local diffeomorphism

$$M//\mathcal{F} \to \mathcal{R}$$

between the stacky leaf space and the classifying stack for Riemannian metrics. So this corresponds to a Riemannian metric *on* the stacky leaf space. One can show in fact that such Riemannian metrics on $M//\mathcal{F}$ are the same as *transverse metrics* on M with respect to the foliation $\mathcal{F}$. (I had Camilo Angulo prove this while coadvising his master class thesis.) Hence, one has that submersions into the classifying stack for Riemannian metrics, classify Riemannian foliations.

There is nothing too special about the sheaf assigning Riemannian metrics, other than that it is a sheaf, so this is quite a general phenomenon. For example, one could equally as well work with sheaf of symplectic forms, and classify foliations with transverse symplectic structure. This general machinery establishes a kind of correspondence between effective étale stacks, and classifying objects for foliations with transverse structure. It's worth mentioning that there is a way to bring leaf-wise structures into the game as well, but I haven't fully worked out the theory yet.

2. Beyond Differential Geometry

After discovering these nice facts about étale *differentiable* stacks, I started wondering if this was really just a special case of a more general theory. For example, can one make similar statements about the Deligne-Mumford stacks in algebraic geometry? How about derived such gadgets, such as the spectral schemes of Lurie? The answer is quite wonderfully YES! This has to do with the beautiful theory of structured ∞-topoi. I know the word topos can be a bit off-putting, even without the ∞-symbol, so let me just say a few words about why this is not nearly as complicated as it sounds.

The basic idea is that for each n, there is the concept of an n-topos, which is basically sheaves of $(n-1)$-groupoids on some site (where I mean homotopy sheaves, e.g., a sheaf of 1-groupoids is what I have been calling a stack). With a bit of "negative thinking" one sees that the concept of n-groupoid makes sense for $n = -1$, so one can define 0-topoi. 0-topoi are basically the same thing as a topological space! (To be honest, they are actually locales, but this is a minor point). In this way, one sees that n-topoi for various n are just *categorifications* of the concept of a topological space.

Now suppose that one considers all smooth manifolds of the form $\mathbb{R}^n$ as locally ringed spaces. In particular, they are locally ringed topoi. One can then look at all locally ringed topoi which can be covered by open subsets of $\mathbb{R}^n$ (regarded as a locally ringed topoi). This is basically how we define n-manifolds. *In fact*, a topological space M is a smooth n-manifold, if and only if $\mathrm{Sh}(M)$ can be made into a locally ringed topos with this property. But when you do this with all 1-topoi instead of just spaces (0-topoi), then you get more than manifolds. The full subcategory of locally ringed topoi you get by gluing $\mathbb{R}^n$'s for various n's is equivalent to the bicategory of étale differentiable stacks. Moreover, one can start considering higher topoi. If we allow 2-topoi in the picture, we get something equivalent to stacks of 2-groupoids on Mfd coming from étale Lie 2-groupoids etc. If you go all the way to infinity topoi, you get a full-fledged theory for étale differentiable ∞-stacks.

This is a very general procedure. For example, if instead of starting with $\mathbb{R}^n$'s, one starts with the collection $\mathrm{Sh}\left(Spec\left(A\right)_{et}\right)$ of the étale topos of each affine scheme (with their appropriate structure sheaf), one arrives at a bicategory equivalent to Deligne-Mumford stacks (with no separation conditions). Similarly for various types of derived schemes. It follows that there should be a general theory of étale stacks, which is about building structured ∞-topoi out of local models. Most of the theorems in these notes generalize to this setting, with the important exception of universal stacks. To have a universal étale stack like $\mathbb{H}$, the local models have to form a set. Even for building ordinary schemes, one cannot get away with any set of affine schemes. However, one can still show e.g. that Deligne-Mumford stacks are characterized by prolongations of stacks on the site of affine schemes and étale maps (with some appropriate size-theoretic conditions). All this seems to be new, and is the subject of a paper I am currently working on.