Introduction to Quasi-variational Inequalities in Hilbert Spaces

Concerning sub-problems: Density and Solvers

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Contents

Density of convex intersections

Motivation

Motivation

- In general, the subproblems associated to solving an elliptic QVI are obtaining the solution $S(y_{n-1})$ to a VI (for some y_{n-1}) where $S(\mathbf{w})$ is the unique solution to

$$\text{Find } y \in \mathbf{K}(\mathbf{w}) : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(\mathbf{w}).$$

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- In image processing, the following class of problems arises

Given $f \in L^2(\Omega)$ and $\alpha : \Omega \to \mathbb{R}$, consider

$$\min \frac{1}{2} \|\text{div}\, \mathbf{p} + f\|^2_{L^2(\Omega)} \quad \text{s.t} \quad \mathbf{p} \in \mathbf{K}, \qquad (\mathbb{P})$$

where

$$\mathbf{K} := \{\mathbf{p} \in H_0(\text{div}) : |\mathbf{p}|_\infty \leq \alpha\},$$

and

$$H_0(\text{div}) := \{w \in L^2(\Omega)^N : \text{div}\, w \in L^2(\Omega) \quad \& \quad \nu \cdot w = 0 \text{ on } \partial\Omega\}.$$

Motivation

The proper choice of α allows to denoise images incredibly well.

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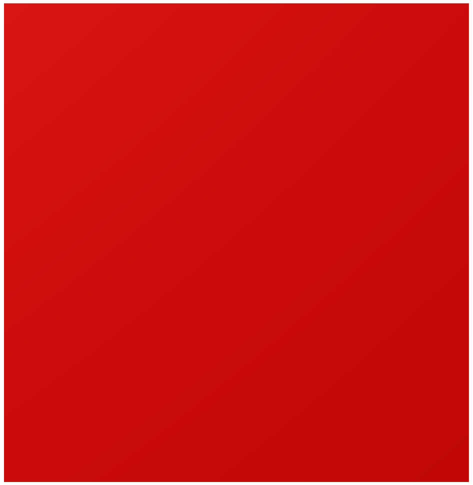
Noisy observation

Reconstruction sequence



Sequence of α

Motivation

In order to approximate the solution to

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We consider the sequence of problems

Let $\gamma_n \to \infty$, and for each $n \in \mathbb{N}$ let $\mathbf{p}_n$ be the solution to

$$\min_{\mathbf{p} \in H_0^1(\Omega)^N} \frac{1}{2} \|\text{div}\, \mathbf{p} + f\|^2_{L^2(\Omega)} + \frac{\gamma_n}{2} \|[|\mathbf{p}| - \alpha]^+\|^2_{L^2(\Omega)} + \frac{1}{2\gamma_n} \|\nabla \mathbf{p}\|_{L^2(\Omega)}. \qquad (\mathbb{P}_n)$$

The term

- $\frac{\gamma_n}{2} \|[|\mathbf{p}| - \alpha]^+\|^2_{L^2(\Omega)}$ is the Moreau-Yosida regularization of $I_{\mathbf{K}}$.
- $\frac{1}{2\gamma_n} \|\nabla \mathbf{p}\|_{L^2(\Omega)}$ is a singular perturbation - lifts from $H_0(\text{div})$ to $H_0^1(\Omega)$.

Motivation

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Q: Does $\mathbf{p}_n$ converges to a solution of $(\mathbb{P})$?

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Well... div $\mathbf{p}_n \to$ div $\mathbf{p}^*$ where $\mathbf{p}^* \in H_0(\text{div})$ solves

$$\min \frac{1}{2}\|\text{div}\,\mathbf{p} + f\|^2_{L^2(\Omega)} \quad \text{s.t} \quad \mathbf{p} \in \overline{\mathbf{K} \cap H_0^1(\Omega)^N}^{H_0(\text{div})}. \qquad (\mathbb{P}^*)$$

Motivation

- Will the regularity of α determines if

$$\overline{\mathbf{K} \cap H_0^1(\Omega)^N}^{H_0(\mathrm{div})} \equiv \mathbf{K},$$

where $\mathbf{K} = \{\mathbf{p} \in H_0(\mathrm{div}) : |\mathbf{p}|_\infty \leq \alpha\}$.

- In the QVI setting, obstacles are implicit (not given a priori), hence understanding density issues becomes of the utmost importance.

Introduction and Further motivations

Consider

- X a Banach space (separable, reflexive)
- $\mathbf{K} \subset X$ a closed, convex, and non-empty.
- $Y \subset X$ a dense subspace, i.e., $\overline{Y}^X = X$

We study the following density question:

Does $\overline{\mathbf{K} \cap Y}^X = \mathbf{K}$ hold true?

The typical example is the initial question one

$$\mathbf{K} = \{f \in X : |f(x)| \leq \alpha(x) \text{ a.e., } x \in \Omega\},$$

where $\Omega \subset \mathbb{R}^N$, $X = W^{1,p}(\Omega)$, and Y is given by smooth maps on Ω, e.g., $C^\infty(\overline{\Omega})$.

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First answer

The dense embedding $Y \hookrightarrow X$ is not enough to guarantee

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Counterexample in [HR(2015)]

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Counterexample in [HR(2015)]

Why do we care?

- Optimization: perturbations and duality.
- Algorithmic and approximation relevance.

- Hintermüller, R., *On the density of classes of closed convex sets with pointwise constraints in Sobolev spaces*, J Math. Anal. Appl., 2015.

Counterexample by Martin Hairer

$\overline{\mathbf{K} \cap Y}^X = \mathbf{K}$ is not necessarily inherited from embeddings!

Let $Y \subset L^2(\Omega)$ be constructed as follows. Define $g \in L^2(\Omega)$ be strictly positive and unbounded on a dense set in Ω, and define $h \in Y$ if $h = zg$ a.e. with $z \in C(\overline{\Omega})$. We endow Y with the norm $|h|_Y := \sup_{x \in \Omega} |z(x)|$.

It follows that Y is dense in $L^2(\Omega)$, and if $\alpha > 0$, then

$$\mathbf{K} = \{f \in L^2(\Omega) : |f(x)| \leq \alpha \text{ a.e., } x \in \Omega\}, \qquad \& \qquad \mathbf{K} \cap Y = \{0\}.$$

A continuous and dense embedding $Y \hookrightarrow X$ is not enough to guarantee

$$\overline{\mathbf{K} \cap Y}^X = \mathbf{K}!$$

Optimization and Regularization Methods

Consider the optimization problem

$$\min F(u) \quad \text{s.t.} \quad u \in \mathbf{K} \qquad (\mathbb{P})$$

- $F : X \to \mathbb{R} \cup \{+\infty\}$ is continuous, seq.-w.l.s.c., and coercive.

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Consider the regularization of $(\mathbb{P})$

$$\min F(u) + R_n(u) \qquad (\mathbb{P}_n)$$

- $R_n : X \to \mathbb{R} \cup \{+\infty\}$ are perturbations of the indicator map $I_{\mathbf{K}}$.
- $I_{\mathbf{K}}(u) = 0$ if $u \in \mathbf{K}$, $I_{\mathbf{K}}(u) = +\infty$ $u \notin \mathbf{K}$.

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Examples for R_n

1. Tikhonov regularization ($\gamma_n \to +\infty$)

$$R_n(u) = I_{\mathbf{K}}(u) + \frac{1}{2\gamma_n}|u|_Y^\alpha.$$

2. Moreau-Yosida-Tikhonov regularization:

$$R_n(u) = \frac{\gamma_n}{2} \inf_{v \in \mathbf{K}} |u - v|_X^2 + \frac{1}{2\gamma_n}|u|_Y^\alpha.$$

3. Conformal disc. $X_n \subset X_{n+1} \subset X$, $\dim(X_n) < \infty$, $\forall n \in \mathbb{N}$

$$R_n(u) = I_{\mathbf{K} \cap X_n}(u), \qquad Y := \cup_{n \in \mathbb{N}} X_n.$$

4. Conf. disc. + Moreau-Yosida reg:

$$R_n(u) = \frac{\gamma_n}{2} \inf_{v \in \mathbf{K}} |u - v|_X^2 + I_{X_n}(u),$$

$Y := \cup_{n \in \mathbb{N}} X_n$.

Optimization and Regularization Methods (Obstacle example)

The obstacle problem

$$\min \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x$$

subject to $u \in \mathbf{K}$,

The regularized obstacle problem

$$\min \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x + R_n(u)$$

over $u \in H_0^1(\Omega)$.

where for some $\alpha : \Omega \to \mathbb{R}$ we consider $\mathbf{K} := \{w \in H_0^1(\Omega) : u \leq \alpha\}$

Examples for R_n

1. **(Tikhonov)** Suppose $\gamma_n \to \infty$, $Y = W_0^{1,p}(\Omega)$ with $p > 2$,

$$R_n(u) = I_{\mathbf{K}}(u) + \frac{1}{2\gamma_n}\int_\Omega |\nabla u|^p \,\mathrm{d}x$$

2. **(Moreau-Yosida-Tikhonov)** regularization:

$$R_n(u) = \frac{\gamma_n}{2}\int_\Omega \max(0, u-\alpha)^2 \,\mathrm{d}x + \frac{1}{2\gamma_n}\int_\Omega |\nabla u|^p \,\mathrm{d}x.$$

Variational Inequalities and singular perturbations

Let X, Y be Hilbert space, with $Y \hookrightarrow X$ dense

- $A : X \to X^*$, and $A_1 : Y \to Y^*$ Lipschitz and strongly monotone.
- $\mathbf{K} \subset X$ closed, convex, and non-empty, and $\mathbf{K} \cap Y$ non-empty.

Consider the sequence of problems

$$\text{Find } u_n \in Y : \left\langle A(u_n) + \frac{1}{\gamma_n} A_1(u_n) - f, v - u_n \right\rangle \geq 0, \qquad \forall v \in \mathbf{K} \cap Y$$

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The limit problem is given by

$$\text{Find } u \in X : \langle A(u) - f, v - u_n \rangle \geq 0, \qquad \forall v \in \overline{\mathbf{K} \cap Y}^X$$

Variational Inequalities and Galerkin methods

Consider the non-monotone Galerkin approximation

$$\text{Find } u_n \in X : \langle A(u_n) - f, v - u_n \rangle \geq 0, \quad \forall v \in \mathbf{K}_n$$

where, in general, $\mathbf{K}_n \not\subset \mathbf{K}$, $\mathbf{K}_{n+1} \not\subset \mathbf{K}_n$.

- Mosco convergence of FEM-discretized $\mathbf{K}_n$ required for consistency+stability.
- Recovery sequences requires interpolation procedure which is only defined on the (supposedly) dense subset $\mathbf{K} \cap Y$ of $\mathbf{K}$ where typically $Y = C^\infty(\overline{\Omega})$ or $Y = C(\overline{\Omega})$.

Gamma-convergence

The Role of Γ-convergence

$$\min F(u) \quad \text{s.t.} \quad u \in \mathbf{K} \qquad (\mathbb{P})$$

$$\min F(u) + R_n(u) \qquad (\mathbb{P}_n)$$

What guarantees that the minimizers of $(\mathbb{P}_n)$ are related to the ones of $(\mathbb{P})$?

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1. If $u_n \to u$ then $G(u) \leq \liminf G_n(u_n)$.
2. For every u, there is a sequence $\{u_n\}$ converging to u such $G(u) \geq \limsup G_n(u_n)$.

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2. For every u, there is a sequence $\{u_n\}$ converging to u such $G(u) \geq \limsup G_n(u_n)$.

Then **minimizers converge to minimizers**: Every cluster point of the sequence of minimizers $\{u_n\}$ to G_n is a minimizer of G.

Density and Γ-convergence

The density property links ($\mathbb{P}$) and ($\mathbb{P}_n$) through Γ-convergence:

Theorem - HRR(2017)

Sufficiency. If $\overline{\mathbf{K} \cap Y}^X = \mathbf{K}$ holds true, then

$$\Gamma - \lim_{n \to \infty} (F + R_n) = F + I_{\mathbf{K}},$$

in both, the weak and strong topology.

Provided that ($\mathbb{P}_n$) admits a minimizer u_n, each weak cluster point of $\{u_n\}$ is a minimizer to ($\mathbb{P}$).

Necessity. In case R_n involves the *Moreau-Yosida regularization* (examples **2** & **4**) then,

$$\Gamma - \lim_{n \to \infty} (F + R_n) = F + I_{\overline{\mathbf{K} \cap Y}^X}.$$

Sufficient conditions

Going back to the initial motivation

Let $\Omega \subset \mathbb{R}^N$ be a domain with smooth boundary $\partial\Omega$ and let $1 \leq p < +\infty$.

Then smooth functions are dense in Sobolev spaces:

$$\overline{C^\infty(\overline{\Omega})}^{W^{1,p}(\Omega)} = W^{1,p}(\Omega),$$

where $W^{1,p}(\Omega)$= Functions in $L^p(\Omega)$ with weak gradients in $L^p(\Omega)^N$.

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Consider an "obstacle" $\alpha : \Omega \to \mathbb{R}$. Does it hold that

$$\overline{\{f \in C^\infty(\overline{\Omega}) : |f| \leq \alpha \text{ a.e.}\}}^{W^{1,p}(\Omega)} = \{f \in W^{1,p}(\Omega) : |f| \leq \alpha \text{ a.e.}\} \ ?$$

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- Is (in this case) the density of $C^\infty(\overline{\Omega})$ into $W^{1,p}(\Omega)$ enough to guarantee the density result?
- Does the regularity of α play a role?

Sufficient conditions on $\alpha : \Omega \to \mathbb{R}$ for density

Let Ω be Lipschitz, $\alpha \in C(\overline{\Omega})$ with $\operatorname{ess\,inf}_{x\in\Omega}\alpha(x) > 0$, consider the the set

$$\mathbf{K}_G(X) := \{f \in X : |(Gf)(x)| \le \alpha(x) \text{ a.e., } x \in \Omega\},$$

where for $1 \le p < \infty$

- $G \in \{\mathrm{id}, \nabla, \mathrm{div}\}$.
- $X \in \{L^p(\Omega), W_0^{1,p}(\Omega), W^{1,p}(\Omega), H_0(\mathrm{div};\Omega)\}$,

where

$$H_0(\mathrm{div}, \Omega) := \{\mathbf{v} \in L^2(\Omega)^N : \mathrm{div}\,\mathbf{v} \in L^2(\Omega), \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},$$

and $W_0^{1,p}(\Omega) = \{v \in L^p(\Omega) : |\nabla v| \in L^p(\Omega)^N, v = 0 \text{ on } \partial\Omega\}$.

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Theorem - HR(2015)-HRR(2017)

1. For $X \in \{L^p(\Omega), W_0^{1,p}(\Omega), H_0(\mathrm{div};\Omega)\}$, and $G = \mathrm{id}$

$$\overline{\mathbf{K}_G(X) \cap C_c^\infty(\Omega)}^X = \mathbf{K}_G(X),$$

2. If $X = W_0^{1,p}(\Omega)$ & $G = \nabla$, or $X = H_0(\mathrm{div};\Omega)$ & $G = \mathrm{div}$, the above holds.
3. If $X = W^{1,p}(\Omega)$, then

$$\overline{\mathbf{K}_{\mathrm{id}}(X) \cap C^\infty(\overline{\Omega})}^X = \mathbf{K}_{\mathrm{id}}(X).$$

- Hintermüller, R., *On the density of classes of closed convex sets with pointwise constraints in Sobolev spaces*, J Math. Anal. Appl., 2015.
- Hintermüller, R., Rösel, *Density of convex intersections and applic.* , Proc. Royal Soc. A, 2017.

Some obstacles are bad!

Are there obstacles $\alpha : \Omega \to \mathbb{R}$ such that the density property of interest does not hold?

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THEOREM - [HRR(2017)]

Let $\Omega \subset \mathbb{R}^N$ with $N \geq 2$, then there exists $\alpha \in W^{1,N}(\Omega) \cap L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega}\alpha(x) > 0$, such that

1. For $1 \leq p \leq \infty$

$$\overline{\mathbf{K}_{\mathrm{id}}(L^p(\Omega)) \cap C(\Omega)}^{L^p(\Omega)} \subsetneq \mathbf{K}_{\mathrm{id}}(L^p(\Omega)).$$

2. For $1 \leq p \leq N$, and for $X \in \{W^{1,p}(\Omega), W_0^{1,p}(\Omega)\}$

$$\overline{\mathbf{K}_{\mathrm{id}}(X) \cap C(\Omega)}^{L^p(\Omega)} \subsetneq \mathbf{K}_{\mathrm{id}}(X).$$

- A highly oscillatory α destroys density!

- Hintermüller, R., Rösel, *Density of convex intersections and applications*, Proc. Royal Soc. A, 2017.

Obstacle arising from PDEs

- Consider a second order differential operator in divergence form:

$$B = \sum_{i,j=1}^{N} -\frac{\partial}{\partial x_i} b_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + b_0(x)$$

where $b_{ij}, b_i, b_0 \in L^\infty(\Omega)$ for $1 \leq i, j \leq N$, the matrix $[b_{ij}(x)]$ is symmetric a.e. in Ω and such that B is uniformly monotone over $H_0^1(\Omega)$, i.e. there exists $\kappa > 0$ such that

$$\langle Bu, u \rangle \geq \kappa |u|^2_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

- We consider

$$\mathbf{K}(X) := \{ f \in X : |f(x)| \leq \alpha(x) \text{ a.e., } x \in \Omega \},$$

suppose that

$$B\alpha = g,$$

where $g \in H^1(\Omega)^*$, and $\langle g, v \rangle \geq 0$ for all non-negative $v \in H_0^1(\Omega)$.

Obstacle arising from PDEs

THEOREM - [HRR(2017)]

Suppose that $\alpha \in H^1(\Omega)$ such that for $B\alpha \geq 0$ in $H^{-1}(\Omega)$, for some B as before. Then

$$\overline{\mathbf{K}(Y \cap H_0^1(\Omega))}^{H_0^1(\Omega)} = \mathbf{K}(H_0^1(\Omega))$$

in the following cases:

1. $\partial\Omega \in C^{0,1}$, $b_{ij} \in C^{0,1}(\Omega)$ or $b_{ij} \in C^1(\Omega)$: $Y = H^2_{\text{loc}}(\Omega)$,
2. $\partial\Omega \in C^{1,1}$ or Ω convex, $b_{ij} \in C^{0,1}(\Omega)$: $Y = H^2(\Omega)$,
3. $\partial\Omega \in C^{0,1}$, $b_{ij}, b_i, b_0 \in C^{m+1}(\Omega)$, $m \in \mathbb{N}_0$: $Y = H^{m+2}_{\text{loc}}(\Omega)$,
4. $\partial\Omega \in C^{m+2}$, $a_{ij}, b_i, c \in C^{m+1}(\overline{\Omega})$, $m \in \mathbb{N}_0$: $Y = H^{m+2}(\Omega)$.

- Hintermüller, R., Rösel, *Density of convex intersections and applications*, Proc. Royal Soc. A, 2017.

A sample of solvers for problems/subproblems

Solvers - A sample of methods

- In the obstacle case, subproblems may reduce to

For $\gamma > 0$, consider

$$F(y) := Ay - f + \gamma(y - \Phi(y))^+ = 0,$$

or

$$F(y) := Ay - f + \gamma(y - \Phi(y_{n-1}))^+ = 0.$$

- In the gradient case,

$$F(y) = Ay - f + \gamma\nabla^*\left((|\nabla y| - \Phi(y_{n-1}))^+ \frac{\nabla y}{|\nabla y|}\right) = 0.$$

Solvers - A sample of methods

Semismooth Newton

In order to solve

$$F(y) = 0,$$

we consider $y_0 \in V$, and the Newton iteration

$$y_{k+1} = y_k - G_F(y_k)^{-1} F(y_k), \qquad k = 0, 1, 2, \dots$$

where $G_F(y)$ is a (presumably invertible) Newton derivative of F, which is defined to satisfy

$$\lim_{h \to 0} \frac{\|F(y+h) - F(y) - G_F(y+h)h\|}{\|h\|} = 0.$$

Solvers - A sample of methods

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Provided $F(y^*) = 0$, $\|G_F(y)^{-1}\| \leq m$ for $y \in N(y^*)$ then $\{y_n\}$ converges **superlinearly** to a solution y^* of $F(y) = 0$ provided $\|y_0 - y^*\|$ is sufficiently small.

From Fréchet to Newton...

Fréchet derivative

$F : D \subset X \to Z$ is called Fréchet differentiable on an open set $U \subset D$ if there exists $G(x) \in \mathcal{L}(X, Z)$ such that, for every $x \in U$,

$$\lim_{|h|_X \to 0} \frac{|F(x+h) - F(x) - G(x)h|_Z}{|h|_X} = 0.$$

From Fr$\acute{e}$chet to Newton...

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Newton derivative

$F : D \subset X \to Z$ is called Newton differentiable on an open set $U \subset D$ if there exists a family of mappings $G : U \to \mathcal{L}(X, Z)$ such that, for every $x \in U$,

$$\lim_{|h|_X \to 0} \frac{|F(x+h) - F(x) - G(x{\color{red}+h})h|_Z}{|h|_X} = 0.$$

The map G is called a Newton derivative of F .

Facts on the Newton derivative

Newton derivatives need not be unique...

Newton derivative - obstacle type case

Denote $F_{\max} : L^q(\Omega) \to L^p(\Omega)$ the pointwise max operator $F_{\max}(x) = \max(0, x)$ and define

$$G_{\max}(x)(s) = \begin{cases} 0, & x(s) < 0; \\ \delta, & x(s) = 0; \\ 1, & x(s) > 0. \end{cases}$$

Then,

- $G_{\max}$ is not in general a N-derivative for $\max(0, \cdot) : L^p(\Omega) \to L^p(\Omega)$ for $1 \leq p \leq \infty$.
- The map $\max(0, \cdot) : L^q(\Omega) \to L^p(\Omega)$ with $1 \leq p < q \leq \infty$ is N-differentiable on $L^q(\Omega)$ and G is an N-derivative. (norm gap phenomenon)

Facts on the Newton derivative

Newton derivative - gradient case

Let $W = W_0^{1,p}(\Omega)$ and $X = W_0^{1,s'}(\Omega)$ where $1/s + 1/s' = 1$ and $3 \leq 3s \leq p < +\infty$ Denote $F_\nabla : W \to X'$ to

$$\langle F_\nabla(y), y\rangle_{X',X} = \int_{\Omega^+(y)} P(\nabla y) \cdot \nabla w \,\mathrm{d}x = \int_{\Omega^+(y)} (|\nabla y| - \varphi)^+ \frac{\nabla y \cdot \nabla w}{|\nabla y|} \,\mathrm{d}x$$

where

$$P(\nabla y) = q(\nabla y) b(\nabla y),$$

with $q(v) = v/|v|$ and $b(z) = (|z| - \varphi)^+$.

Facts on the Newton derivative

Newton derivative - gradient case

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where

$$P(\nabla y) = q(\nabla y) b(\nabla y),$$

with $q(v) = v/|v|$ and $b(z) = (|z| - \varphi)^+$.

Then, $G_\nabla(y) : W \to X'$ given by

$$\langle G_\nabla(y)v, w\rangle_{X',X} = \int_{\Omega^+(y)} (G_P(\nabla y)\nabla v) \cdot \nabla w \,\mathrm{d}x,$$

for all $y, v \in W$ and $w \in X$ is a Newton derivative of F_∇.

Newton derivative - gradient case

Let $P(\nabla y) = q(\nabla y)b(\nabla y)$ with $q(v) = v/|v|$ and $b(z) = (|z| - \varphi)^+$. Then, $G_\nabla(y) : W \to X'$ given by

$$\langle G_\nabla(y)v, w\rangle_{X',X} = \int_{\Omega^+(y)} (G_P(\nabla y)\nabla v) \cdot \nabla w \, \mathrm{d}x,$$

for all $y, v \in W$ and $w \in X$ is a Newton derivative of F_∇, where $G_P(y) : L^p(\Omega)^n \to L^s(\Omega)^n$ with $3 \le 3s \le p < \infty$ given by

$$G_P(y) = q(y)G_b(y) + b(y)Q(y),$$

is a Newton derivative of P.

Here $G_b \in \mathcal{L}(L^p(\Omega)^n, L^s(\Omega))$ for $1 \le \hat{s} < \hat{p} \le \infty$ given by

$$G_b(y) = G_{\max}(|y| - \varphi)y^T/|y|$$

is the Newton derivative of $b : L^{\hat{p}}(\Omega)^n \to L^{\hat{s}}(\Omega)$, and Q is given by

$$Q(y) = \frac{1}{|y|}\left(\mathrm{id} - \frac{yy^T}{|y|^2}\right)$$

An augmented Lagrangian methods from Kanzow and Steck

Let $A : V \to V'$ and $f \in V'$ for some (real) Hilbert space V. Consider

$$\text{Find } y \in \mathbf{K}(y) : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(y) \tag{QVI}$$

where for $G(w, z) = \Phi(w) - \Psi(Gz)$, we take

$$\mathbf{K}(w) := \{z \in V : G(w, z) \geq 0\}.$$

The following Lagrangian can be considered if Ψ is smooth

$$\mathcal{L}_\rho(y, \lambda)h = \langle A(y) - f, h \rangle + \rho \left(G(y, y) + \frac{\lambda}{\rho} - P_{[0,+\infty)} \left(G(y, y) + \frac{\lambda}{\rho} \right), \Psi'(Gy)Gh \right),$$

and provided that you have a good solver for

$$\mathcal{L}_\rho(y, \lambda) = 0,$$

and augmented Lagrangian algorithm can be considered with function space convergent properties and applicability to a wide variety of examples!

Thanks for your attention!

Thanks Christian, Daniel, Silke, and all people involved in the organization!

The speakers felt at home in Würzburg!