Introduction to Quasi-variational Inequalities in Hilbert Spaces

Time dependent (parabolic) problems

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Septemper 2019

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The parabolic QVI problem

From parabolic problems to parabolic (evolutionary) VIs

Let V be a Hilbert space, $A : V \to V'$, and $f : (0,T) \to V'$. Consider

Find $u \in L^2(0,T;V)$ with $u(0) = u_0$, and $\partial_t u \in L^2(0,T;V')$ such that

$$\langle \partial_t u + A(u) - f, v\rangle = 0,$$

for all $v \in L^2(0,T;V)$.

- No need to explain the importance for parabolic problems.

- $V \in \{H_0^1(\Omega), H^1(\Omega), L^2(\Omega), \ldots\}$, and part of a Gelfand triple (V, H, V').
- A is Lipschitz continuous and strongly monotone, i.e.,

$$\langle A(u) - A(v), u - v\rangle \geq c\|u - v\|_V^2.$$

- $f \in L^2(0,T;V')$.

- R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, AMS, 1997.

From parabolic problems to parabolic (evolutionary) VIs

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- No need to explain the importance for parabolic problems.

Not all dynamics in applications arise from parabolic equations:

- In many cases the state u satisfies constraints, i.e.,

$$u(t) \in \mathbf{K}(t),$$

f.a.a. $t \in (0,T)$, where, $\mathbf{K}(t)$ is a closed, convex, and non-empty subset of V.

From parabolic (evolutionary) VIs to QVIs

Let V be a Hilbert space, $A : V \to V'$, and $f : (0,T) \to V'$. Consider

Find $u \in L^2(0,T;V)$ with $u(0) = u_0$, and $\partial_t u \in L^2(0,T;V')$ such that $u(t) \in \mathbf{K}(t)$ a.e. and

$$\langle \partial_t u + A(u) - f, v - u \rangle \geq 0,$$

for all $v \in L^2(0,T;V)$ with $v(t) \in \mathbf{K}(t)$ a.e..

- The behaviour of $t \mapsto \mathbf{K}(t)$ plays a significant role now! The cases

$$\text{i.} \quad \mathbf{K}(t) \subset \mathbf{K}(t+\Delta t) \qquad\qquad \text{ii.} \quad \mathbf{K}(t) \supset \mathbf{K}(t+\Delta t),$$

have completely different difficulties.

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Not all dynamics in applications arise from parabolic (evolutionary) VIs :

- In many cases the constraint $\mathbf{K}$ depends also on the state u i.e.,

$$u(t) \in \mathbf{K}(u(t)) \qquad \text{f.a.a.} \quad t \in (0,T).$$

Parabolic (evolutionary) QVIs

Let V be a Hilbert space, $A : V \to V'$ a monotone coercive operator, and consider

Find $u \in L^2(0,T;V)$ with $u(0) = u_0$, and $\partial_t u \in L^2(0,T;V')$ such that $u(t) \in \mathbf{K}(u(t))$ a.e. and

$$\langle \partial_t u + A(u) - f, v - u \rangle \geq 0,$$

for all $v \in L^2(0,T;V)$ with $v(t) \in \mathbf{K}(u(t))$ a.e..

Many contributors Adly, Aubin, Aussel, Barrett, Bensoussan, Bergounioux, Biroli, Caffarelli, Facchinei, Friedman, Frehse, Fukao, Fukushima, Gwinner, Hanouzet, Hintermüller, Joly, Kano, Kenmochi, Lions, Mignot, Mordukhovich, Mosco, Murase, Outrata, Pang, Prigozhin, Rockafellar, Rodrigues, Santos, Stefanelli, Tartar, Yousept,

Main source of difficulties: The constraint $\mathbf{K}(\cdot)$ depends on the state itself.

Applications

Granular Materials: angle of repose

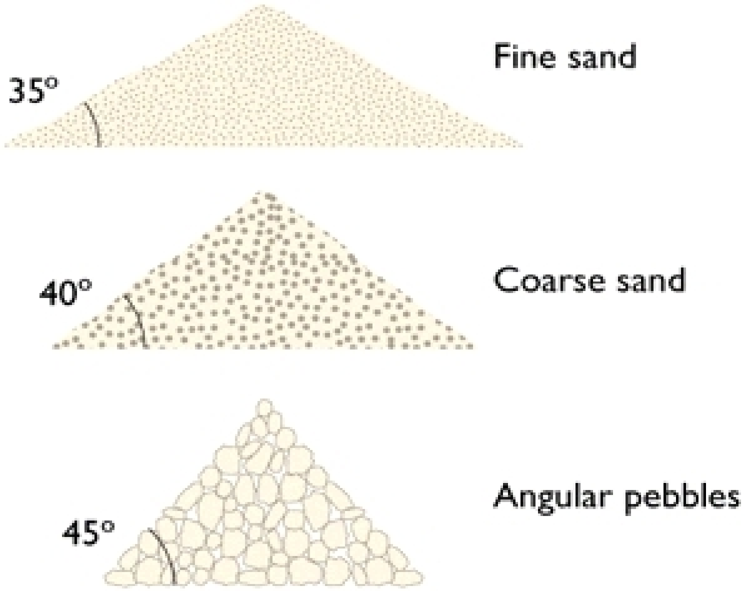

Figure: Gravel pouring from a “point” source (left). Angle of repose of several materials (right)

If $u(t,x)$ denotes the surface of the growing pile of a granular material (at time t) and θ is its angle of repose, then

$$|\nabla u(t,x)| \leq \tan(\theta),$$

where ∇ is the spatial gradient.

Non-flat Supporting Surface u_0

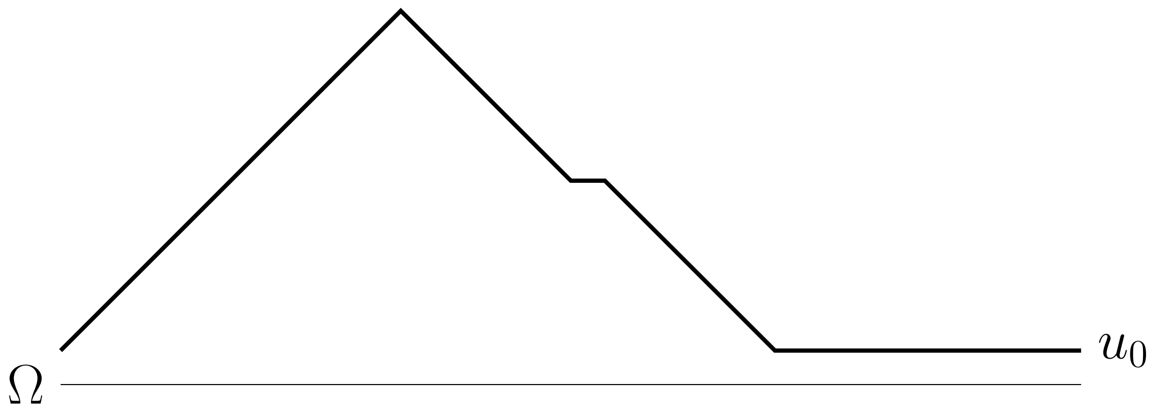

Figure: Sand distribution on a pile

- $u_0(x)$ – solid supporting surface
- $f(t, x)$ determines at which rate a granular incompressible material is poured onto a solid surface
- θ – angle of repose of the granular material
- $u(t, x)$ – surface (solid surface+distributed material)

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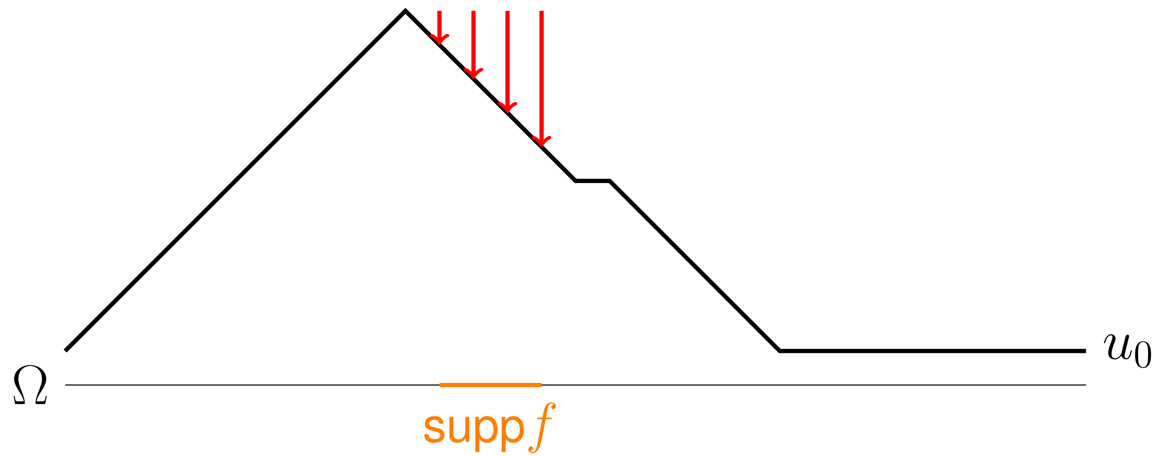

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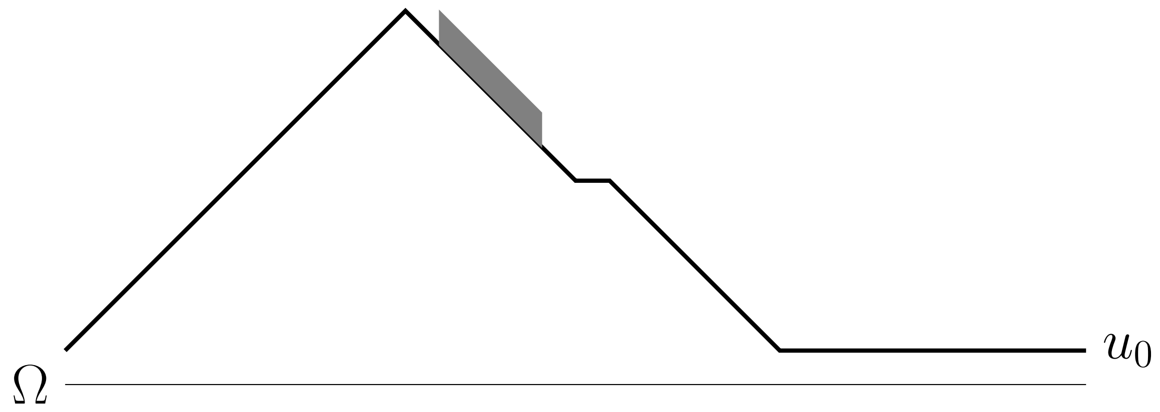

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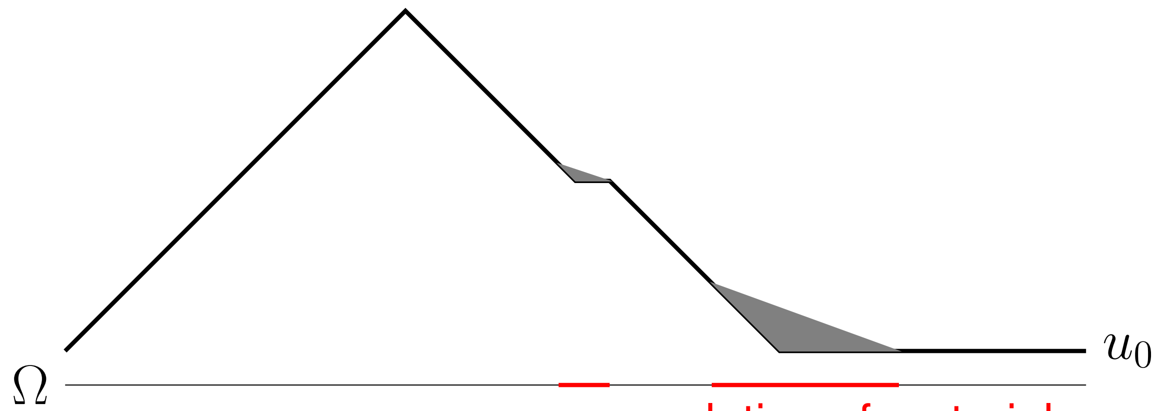

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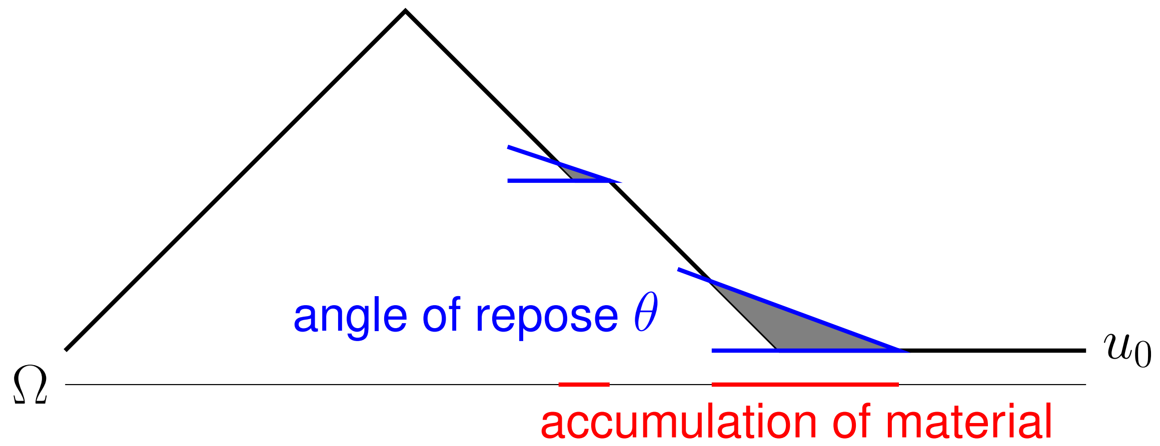

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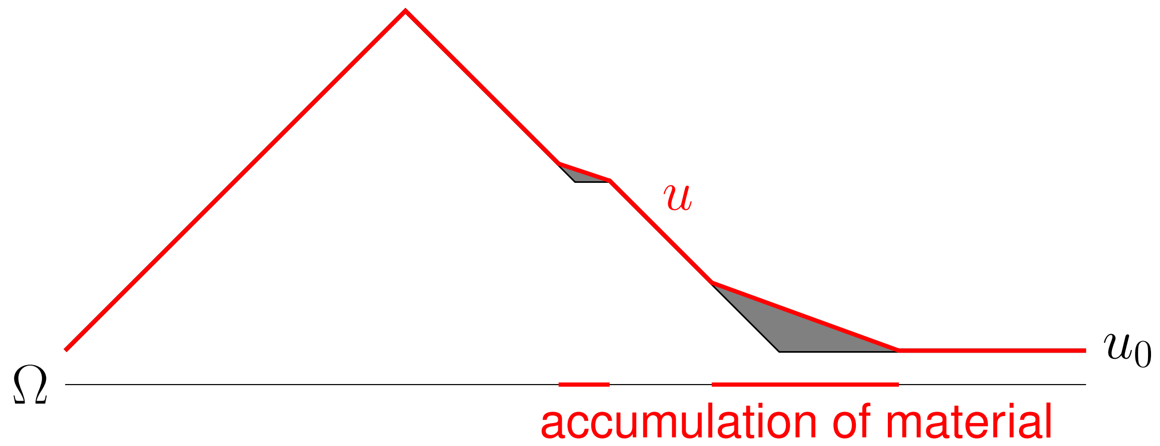

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Growing of Sandpiles ([Prigozhin(1986)])

If the intensity of the source of material being poured onto the pile is given by $(t, x) \mapsto f(t, x) \geq 0$, u satisfies $u(0) = u_0$ and

$$u \in \mathbf{K}(u) : \langle \partial_t u - f, v - u \rangle \geq 0, \quad \forall v \in \mathbf{K}(u),$$

where $\mathbf{K}(u) = \{w \in H_0^1(\Omega) : |\nabla w| \leq \Phi(u)$ a.e.$\}$, and

$$\Phi(u)(t, x) = \begin{cases} \tan(\theta), & u(t, x) > u_0(x); \\ \max(\tan(\theta), |\nabla u_0(x)|), & \text{otherwise.} \end{cases}$$

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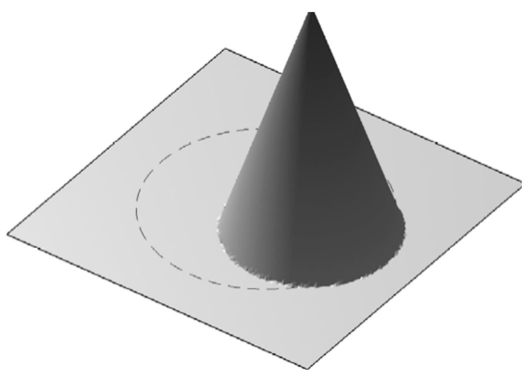
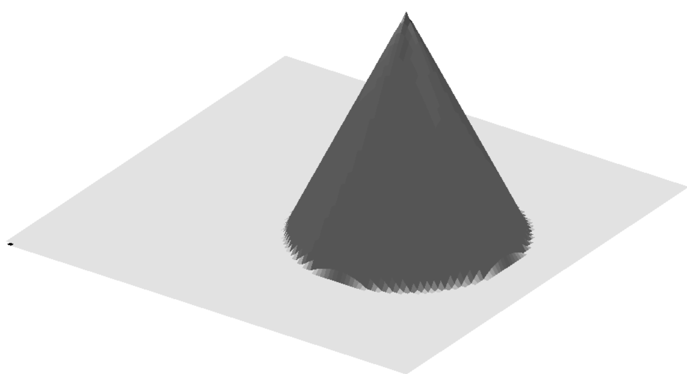
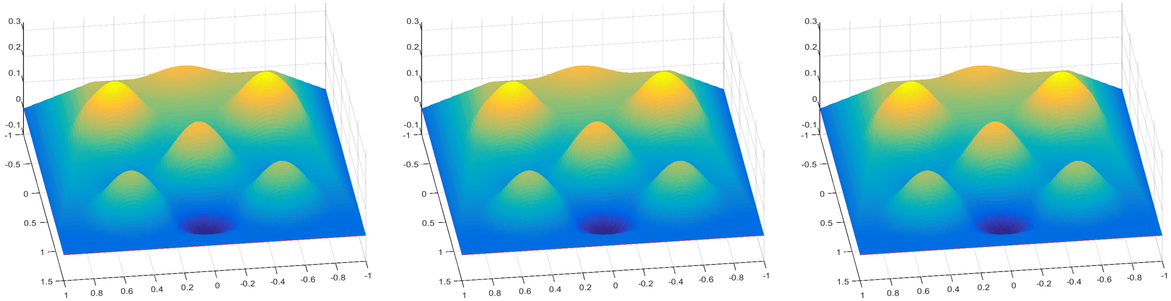
The model describes well accumulation of material on steep structures





Growing of Sandpiles - One addition ($\theta \downarrow 0$ case)

- **Determination of river/lake networks via the accumulation of granular materials.** ([Barrett,Prigozhin]) Water is considered as a cohensionless material with zero angle of repose.



Material accumulation as angle of repose decreases.

As $\theta \downarrow 0$,

$$(t, x) \mapsto u(t, x)$$

resembles a fluid.

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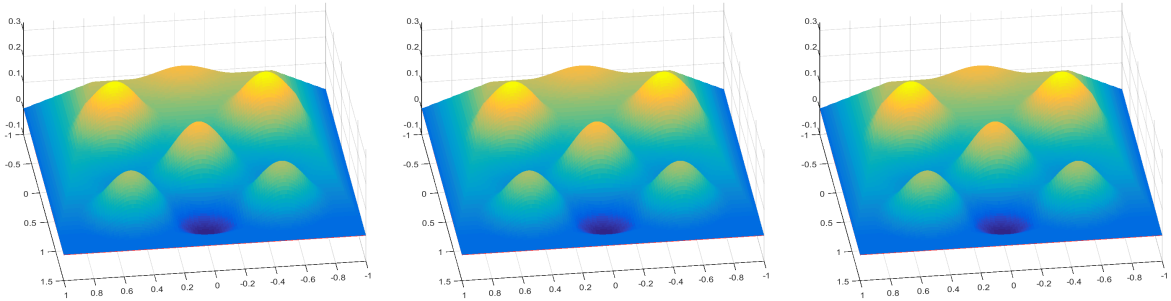

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Nonlinear and nonlocal effects when $\theta = 0$.

In the inequality level, we have a new nonlinear term $\Theta(u)$ accounting for effects of permeability of soil, saturation, etc...

The modified Prigozhin model, when $0 < \theta \ll 1$, is

$$\text{Find } u \in \mathbf{K}(u) : \langle \partial_t u - \Theta(u) - f, v - u \rangle \geq 0, \quad \forall v \in \mathbf{K}(u),$$

with $u(0) = u_0$, where $\mathbf{K}(u) = \{w \in H_0^1(\Omega) : |\nabla w| \leq \Phi(u) \text{ a.e.}\}$.

Growing of Sandpiles - Another addition

Large piles are more complex!

Recently, It has been discovered that the angle of repose θ is actually a gravity dependent quantity (see [Kleinhans et. al (2011)]) and hence it should be taken as an increasing function of the height of the pile:

$$u_0 \equiv 0 \quad \Longrightarrow \quad u \mapsto \Phi(u) \quad \text{is increasing.}$$

- M. G. Kleinhans, H. Markies, S. J. de Vet, A. C. in 't Veld, and F. N. Postema, *Static and dynamic angles of repose in loose granular materials under reduced gravity*, Journal of Geophysical Research (2011)

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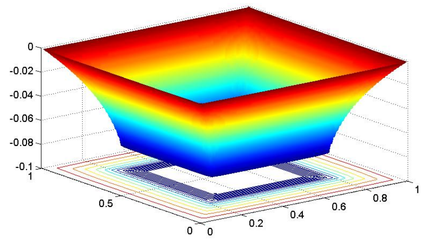
For $u_0 \equiv 0$, the modified Prigozhin model is

$$\text{Find } u \in \mathbf{K}(u) : \langle \partial_t u - f, v - u \rangle \geq 0, \quad \forall v \in \mathbf{K}(u),$$

with $u(0) = u_0$, where $\mathbf{K}(u) = \{w \in H_0^1(\Omega) : |\nabla w| \leq \Phi(u) \text{ a.e.}\}$, and

$$\Phi(u) = \alpha + \beta u.$$

A dissipative application - Type-II superconductors



- **Stationary Magnetization of a superconductor**([Prigozhin, Rodrigues, Yousept]) : Determination of the magnetic field.

The evolution of the z-component of the magnetic field u in a type-II superconductor under the influence of an external magnetic field b can be the described by the following QVI (in adimensional form)

The QVI that describes u is given by

$$\text{Find } \; u \in \mathbf{K}(u) : \langle \partial_t u - \Delta u - f, v - u \rangle \geq 0, \quad \forall v \in \mathbf{K}(u),$$

with $u(0) = u_0$, $\mathbf{K}(u) = \{w \in H_0^1(\Omega) : |\nabla w| \leq \Phi(u) \text{a.e.}\}$, and $f(t) = \partial_t b(t)$.

The operator Φ is a Nemytskii operator induced by a function $\phi : \mathbb{R} \to \mathbb{R}$ that is increasing on some interval $[x_1, x_2]$ (not globally though).

A parabolic QVI with an extra non-linearity

Evolutionary QVIs with gradient constraints

Problem (P) : Find $u \in L^2(0,T;H_0^1(\Omega))$, with $u(0) = u_0 \in H_0^1(\Omega)$ and $u(t) \in \mathbf{K}(\Phi(u)(t))$ a.e. such that

$$\langle \partial_t u + Au - \Theta(u) - f, v - u \rangle \geq 0,$$

for every $v \in L^2(0,T;H_0^1(\Omega))$, with $v(t) \in \mathbf{K}(\Phi(u)(t))$ a.e.

For a non-negative ϕ, $\mathbf{K}(\phi) \subset H_0^1(\Omega)$ is defined as

$$\mathbf{K}(\phi) := \{v \in H_0^1(\Omega) : |\nabla v| \leq \phi \text{ a.e. in } \Omega\}.$$

Notice:

- Θ is a nonlinear term - it may be responsible for finite-time blow up.
- We focus on the gradient constraint case, but all results follows identically for

$$\mathbf{K}(\phi) := \{v \in H_0^1(\Omega) : v \leq \phi \text{ a.e. in } \Omega\}.$$

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Features of the problem:

- The main actors are Θ and Φ.
- Difficult to develop solution algorithms.
- Hard to obtain qualitative properties (e.g. non-decreasing solutions).

- M. Hintermüller, C. N. R., N. Strogies, *Dissipative and Non-dissipative Evolutionary Quasi-variational Inequalities with Gradient Constraints*, Set-Valued Var. Anal, 2018.

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We differentiate between two problems:

Problem (P_0):

Solve problem (P) with $A \equiv 0$ (**a non-dissipative problem**) .

Problem (P_1):

Solve problem (P) when $A \not\equiv 0$ is a monotone operator (**a dissipative problem**).

A VI semi-discretization approach

Let $N \in \mathbb{N}$, $k := T/N$, $t_n := nk$ and $\mathrm{I}_n := [t_{n-1}^N, t_n^N)$ with $n = 0, 1, \ldots, N$.

Problem (P^N) : Find $\{u_n^N\}_{n=0}^N$ with $u_0^N = u_0$, $u_n^N \in \mathbf{K}(\Phi(u_{n-1}^N))$, such

$$\left\langle \frac{u_n^N - u_{n-1}^N}{k} + A(u_n^N) - \Theta(u_{n-1}^N) - f_n^N, v - u_n^N \right\rangle \geq 0,$$

for all $v \in \mathbf{K}(\Phi(u_{n-1}^N))$ with

$$f^N = \sum_{n=1}^{N} f_n^N \chi_{[t_{n-1}^N, t_n^N)} \qquad \text{and} \qquad f_n^N = \frac{1}{k} \int_{t_{n-1}^N}^{t_n^N} f(t)\, \mathrm{d}t.$$

- We evaluate Θ and Φ on the previous time step.
- Each n-subproblem is amenable for computational implementation: equivalent to an optimization problem (if A symmetric).

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- Each n-subproblem is amenable for computational implementation: equivalent to an optimization problem (if A symmetric).

Again, we differentiate between two problems:
Problem (P_0^N): $A \equiv 0$ (**the non-dissipative problem**).
Problem (P_1^N): $A \not\equiv 0$ is a monotone operator (**the dissipative problem**).

The Non-Dissipative ($A \equiv 0$) case
Problem (P_0).

Assumptions

i. $f \in L^\infty(0,T;L^2(\Omega))$ is non-negative, i.e., $f(t) \geq 0$ a.e. in Ω, for a.e. $t \in (0,T)$.

ii. The initial condition $u_0 \in H_0^1(\Omega)$ satisfies $|\nabla u_0| \leq \Phi(u_0)$ a.e. in Ω.

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ii. The initial condition $u_0 \in H_0^1(\Omega)$ satisfies $|\nabla u_0| \leq \Phi(u_0)$ a.e. in Ω.

iii. $\Theta : L^2(\Omega) \to L^2(\Omega)$ is uniformly continuous and satisfies $\Theta(v) \geq 0$ a.e. if $v \geq u_0$ a.e. in Ω, for a.e. $t \in [0,T]$. It is further assumed that Θ has α-order of growth:

$$\exists\, \alpha > 0, L_\Theta > 0 : \quad |\Theta(v)|_{L^2(\Omega)} \leq L_\Theta |v|^\alpha_{L^2(\Omega)}, \quad \forall v \in L^2(\Omega).$$

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$$\exists\, \alpha > 0, L_\Theta > 0 : \quad |\Theta(v)|_{L^2(\Omega)} \leq L_\Theta |v|^\alpha_{L^2(\Omega)}, \quad \forall v \in L^2(\Omega).$$

iv. The operator $\Phi : L^2(\Omega) \to L^\infty(\Omega)$ is uniformly continuous and $\Phi(v) \geq \nu > 0$ a.e. in Ω and all $v \in L^2(\Omega)$. We also assume that Φ is non-decreasing:

$$u_0 \leq v_1 \leq v_2 \text{ a.e.} \quad \Longrightarrow \quad \Phi(v_1) \leq \Phi(v_2) \text{ a.e.}.$$

Theorem ([Hintermüller-R.-Strogies(2018)]) Let $\alpha \in [0,1]$. Then there exists a solution u^* to problem (P_0) that is **non-decreasing** and satisfies:

$$u^* \in L^\infty(0,T;W_0^{1,\infty}(\Omega)) \cap C^{0,1}([0,T];L^2(\Omega)), \qquad \partial_t u^* \in L^\infty(0,T;L^2(\Omega)).$$

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The sequence $\{\tilde{u}^N\}$ defined by

$$\tilde{u}^N(t) = u_0 + \int_0^t \sum_{n=1}^N \frac{u_n^N - u_{n-1}^N}{k} \chi_{[t_{n-1},t_n)}(s)\,\mathrm{d}s,$$

where $\{u_n^N\}_{n=0}^N$ solves (P_0^N), satisfies

$$\tilde{u}^N \to u^* \text{ in } C([0,T];L^2(\Omega)) \qquad \text{and} \qquad \partial_t \tilde{u}^N \rightharpoonup \partial_t u^* \text{ in } L^2(0,T;L^2(\Omega)),$$

along a subsequence.

Theorem ([Hintermüller-R.-Strogies(2018)]) Let $\alpha \in [0,1]$. Then there exists a solution u^* to problem (P_0) that is **non-decreasing** and satisfies:

$$u^* \in L^\infty(0,T;W_0^{1,\infty}(\Omega)) \cap C^{0,1}([0,T];L^2(\Omega)), \qquad \partial_t u^* \in L^\infty(0,T;L^2(\Omega)).$$

The sequence $\{\tilde{u}^N\}$ defined by

$$\tilde{u}^N(t) = u_0 + \int_0^t \sum_{n=1}^N \frac{u_n^N - u_{n-1}^N}{k} \chi_{[t_{n-1},t_n)}(s)\,\mathrm{d}s,$$

where $\{u_n^N\}_{n=0}^N$ solves (P_0^N), satisfies

$$\tilde{u}^N \to u^* \text{ in } C([0,T];L^2(\Omega)) \qquad \text{and} \qquad \partial_t \tilde{u}^N \rightharpoonup \partial_t u^* \text{ in } L^2(0,T;L^2(\Omega)),$$

along a subsequence.

Furthermore, if $\alpha > 1$, then the same holds true provided that

$$|u_0|_{L^2(\Omega)} + L_\Theta |u_0|^\alpha_{L^2(\Omega)} + T^{1/2}|f|_{L^2(0,T;L^2(\Omega))} < \frac{1}{((\alpha-1)L_\Theta T)^{\frac{1}{\alpha-1}}}.$$

The dissipative ($A \neq 0$) case
Problem (P_1).

Assumptions

i. The operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is of the form

$$\langle Av, w \rangle = \sum_{n=1}^{N} a_n \int_\Omega \frac{\partial v}{\partial x_n} \frac{\partial w}{\partial x_n} \, \mathrm{d}x \quad \forall v, w \in H_0^1(\Omega)$$

with $a_n \geq a > 0$, $a_n \in \mathbb{R}$ for $n = 1, 2, \ldots, N$.

ii. $f \in L^\infty(0, T; \mathbb{R})$ is non-decreasing.

iii. $u_0 \in H_0^1(\Omega)$ satisfies $A(u_0) \in L^2(\Omega)$, $|\nabla u_0| \leq \Phi(u_0)$ and

$$A(u_0) \leq \Theta(u_0) + \frac{1}{\epsilon} \int_0^\epsilon f(t) \, \mathrm{d}t, \qquad \forall \epsilon > 0 \quad \text{sufficiently small.}$$

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iv. $\Theta : L^2(\Omega) \to \mathbb{R}$ is uniformly continuous, non-decreasing:

$$u_0 \leq v_1 \leq v_2 \text{ a.e.} \quad \Longrightarrow \quad \Theta(v_1) \leq \Theta(v_2) \text{ a.e.},$$

and has α-order of growth:

$$\exists\, \alpha > 0, L_\Theta > 0 : \quad |\Theta(v)| \leq L_\Theta |v|^\alpha_{L^2(\Omega)}, \quad \forall v \in L^2(\Omega).$$

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Theorem ([Hintermüller-R.-Strogies(2018)]) Let $\alpha \in [0,1]$, then there is a solution u^* to problem (P_1) such that

$$u^* \in L^\infty(0,T;W_0^{1,\infty}(\Omega)) \cap C^{0,1}([0,T];L^2(\Omega)), \quad \text{and} \quad \partial_t u^* \in L^\infty(0,T;L^2(\Omega)).$$

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3. An algorithm and further numerical results

A simple variable splitting approach

Suppose that u_{n-1}^N is given, how do we approximate u_n^N?

Recall that $u_n^N \in \mathbf{K}(\Phi(u_{n-1}^N))$, and

$$\left\langle \frac{u_n^N - u_{n-1}^N}{k} + A(u_n^N) - \Theta(u_{n-1}^N) - f_n^N, v - u_n^N \right\rangle \geq 0,$$

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for all $v \in \mathbf{K}(\Phi(u_{n-1}^N))$.

Let γ be "sufficiently large" and consider

Problem (P_γ^n) :

$$\min \mathcal{J}_{n,\gamma}^N(u,p) := \tfrac{1}{2k}|u - u_{n-1}^N|_{L^2(\Omega)}^2 + \tfrac{1}{2}\langle Au, u\rangle - (\Theta(u_{n-1}^N) + f(t_{n-1}), u) + \tfrac{\gamma}{2}|\nabla u - p|_{L^2(\Omega)^\ell}^2$$

over $(u,p) \in H_0^1(\Omega) \times L^2(\Omega)^\ell$

s.t. $|p| \leq \Phi(u_{n-1}^N)$.

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over $(u,p) \in H_0^1(\Omega) \times L^2(\Omega)^\ell$

s.t. $|p| \leq \Phi(u_{n-1}^N)$.

Algorithm

Data: $n \in \mathbb{N}, k, \gamma \in \mathbb{R}^+, u_{n-1}^N \in L^2(\Omega)$.

1. Choose $u^{(0)} \in L^2(\Omega)$ and set $l = 0$.

2. repeat

3. Compute $p^{(l+1)} = \text{argmin}_{p \in L^2(\Omega)^\ell} |p - \nabla u^{(l)}|_{L^2(\Omega)^\ell}^2 + I_{|q| \leq \Phi(u_{n-1}^N)}(p)$.

4. Compute $u^{(l+1)} = \text{argmin}_{u \in H_0^1(\Omega)} \mathcal{J}_{n,\gamma}^N(u, p^{(l)})$.

5. Set $l = l + 1$.

6. until some stopping rule is satisfied.

Magnetic field for a type-II superconductor (different penalty γ parameters)

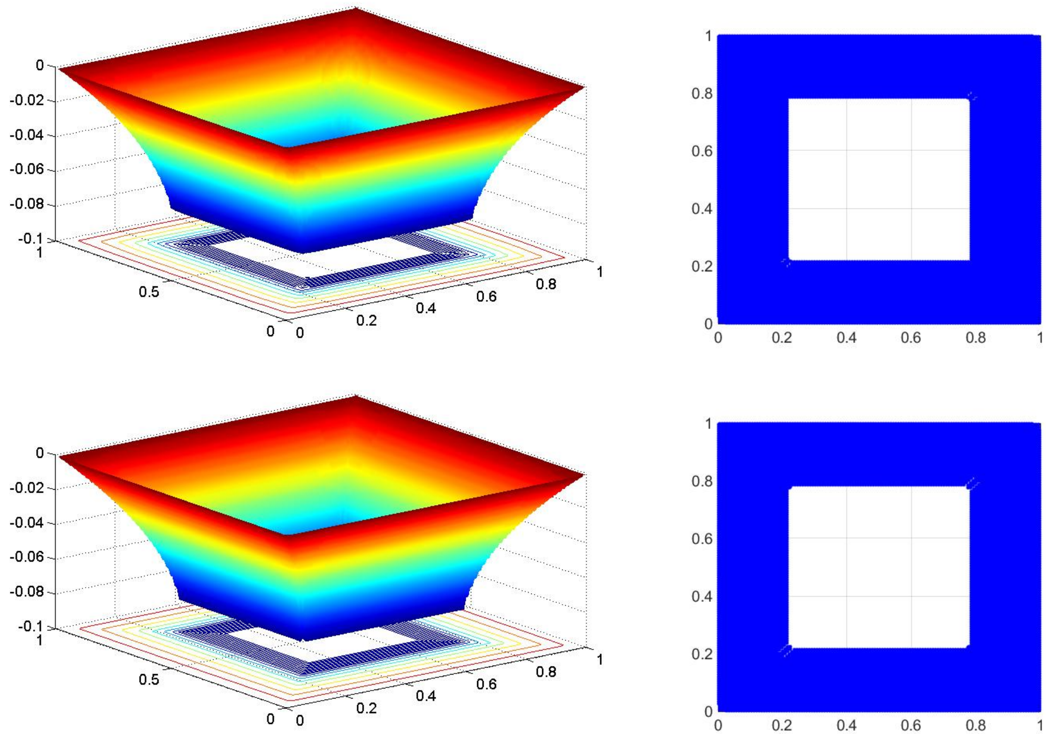

Figure: The final state and active set for $\gamma = 10$ (above), and for $\gamma = 100$ (below)

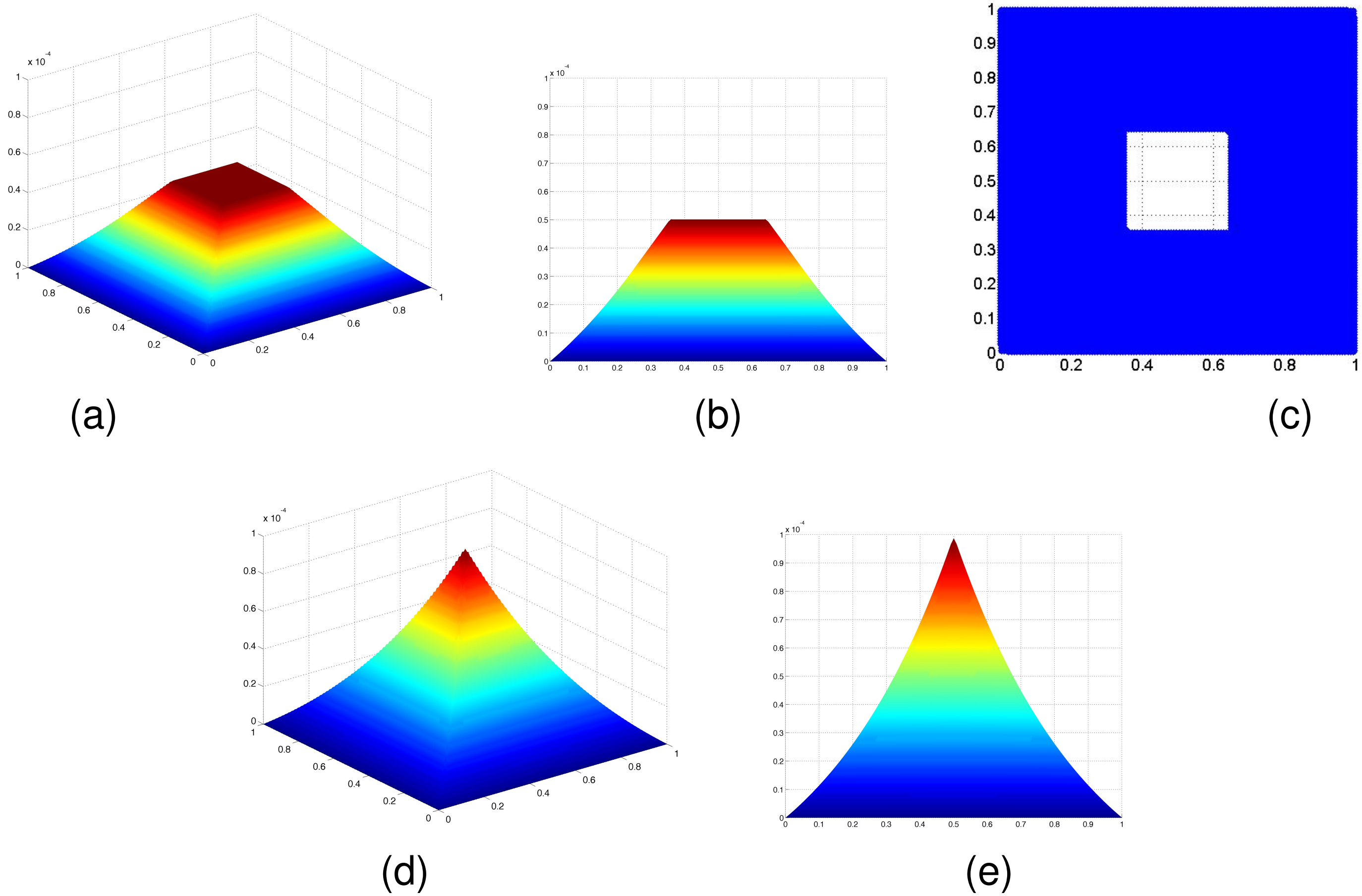

Figure: The state $u(t)$ at time $t = 5 \times 10^{-5}$ is depicted in figures (a), (b) and at $t = 10^{-3}$ in (d) and (e). The active set $\mathcal{A}(t)$ at $t = 5 \times 10^{-5}$ is given in (c)

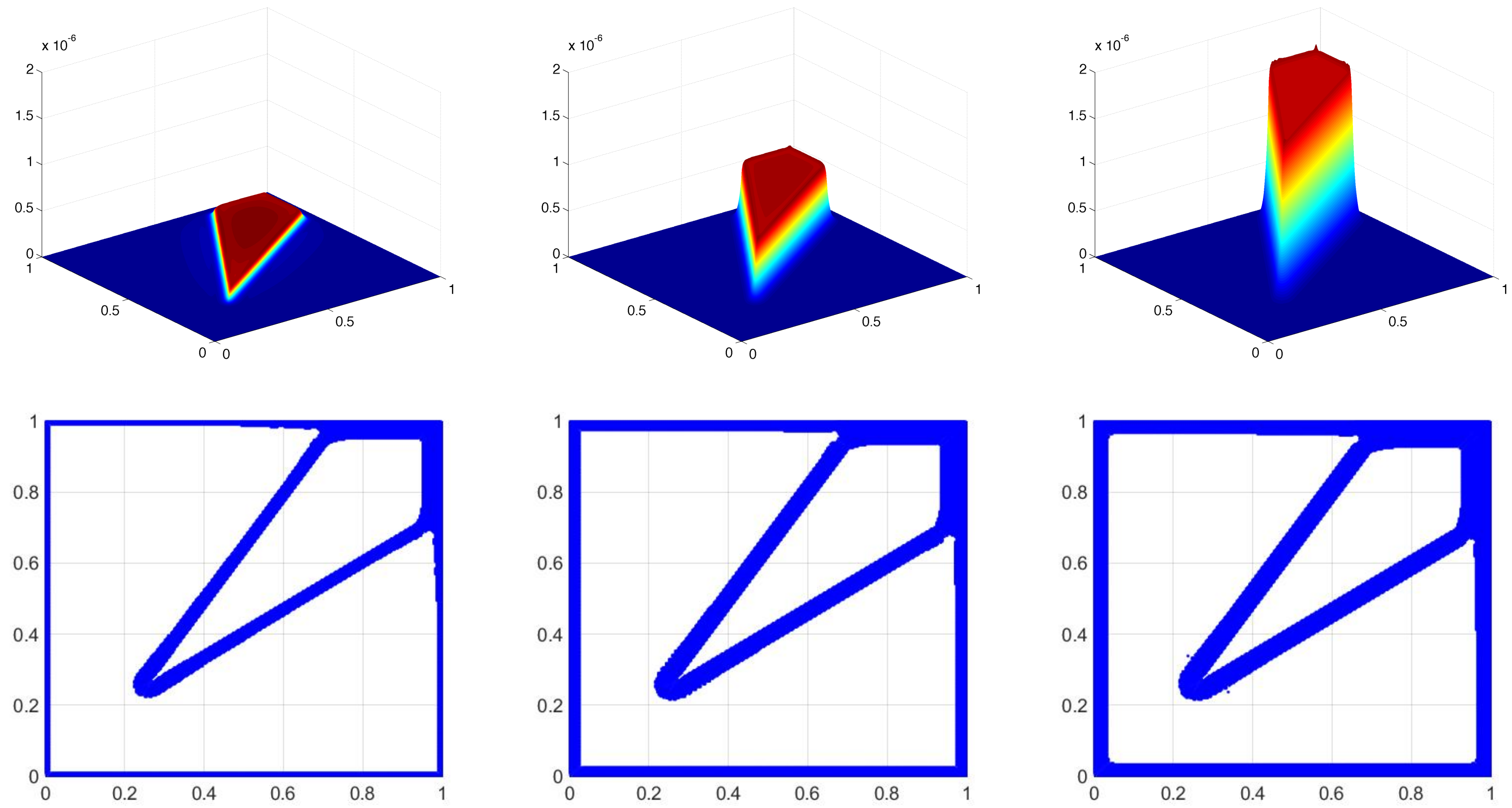

Figure: The state $u(t)$ at times $t = 10^{-7}, 5 \cdot 10^{-7}, 10^{-6}$ is depicted in first row. The corresponding active sets $\mathcal{A}(t)$ at those same times are given in the second row.

Thanks for your attention!