Introduction to Quasi-variational Inequalities in Hilbert Spaces

Exploiting order

C. N. Rautenberg
Department of Mathematical Sciences
George Mason University
Fairfax, VA, U.S.

September 2019

Contents

The obstacle elliptic QVI problem

The class of elliptic QVIs

Let $A : V \to V'$ and $f \in V'$ for some (real) Hilbert space V. Consider

$$\text{Find } y \in \mathbf{K}(y) : \langle Ay - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(y) \tag{QVI}$$

where

$$\mathbf{K}(w) := \{z \in V : z \leq \Phi(w)\}.$$

Objectives/Goals:

- In general there are multiple solutions. The solution set $\mathbf{Q}(f)$ might be of *any* cardinality. However, we want to understand stability and directional differentiability properties of
$$f \mapsto \mathbf{Q}(f).$$
- Further understanding on the structure of $\mathbf{Q}(f)$ is needed.

Assumptions on V, A, and $\mathbf{K}$

- Gelfand triple of Hilbert spaces (V, H, V'), and $L^\infty(\Omega) \hookrightarrow H$. Order induced in H by a closed convex cone, with $|v^+|_V \leq C|v|_V$ for some $C > 0$ and all $v \in V$.

- The map $A : V \to V'$ is linear, uniformly monotone,

$$\langle Au, u \rangle \geq c|u|_V^2, \quad \forall u \in V, \qquad (c > 0)$$

  and that for all $v \in V$, we have

$$\langle Av^-, v^+ \rangle \leq 0.$$

- The map $\mathbf{K}$ is defined as

$$\mathbf{K}(y) = \{v \in V : v \leq \Phi(y)\},$$

  where the map $\Phi : V \to V$ is **increasing**:

$$v \leq w \implies \Phi(v) \leq \Phi(w).$$

Assumptions on V, A, and $\mathbf{K}$ - Examples

The typical setting is given by

- $(V, H, V') = (H_0^1(\Omega), L^2(\Omega), H^{-1}(\Omega))$. Order induced in $L^2(\Omega)$ is via $L^2_+(\Omega)$.

- The map $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is given

$$\langle Av, w \rangle = \int_\Omega \left(\sum_{i,j} a_{ij}(x) \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_i} + \sum_i a_i(x) \int_\Omega \frac{\partial v}{\partial x_j} w + a_0(x) vw \right) \, \mathrm{d}x,$$

with usual assumptions over coefficients. Also fractional powers A^s for $s \in (0,1)$ are suitable.

- $\Phi : H_0^1(\Omega) \to H_0^1(\Omega)$ is
  - A *superposition operator*, i.e., $\Phi(y)(x) = \varphi(y(x))$ for some φ.
  - A *solution operator coming from a PDE*, e.g., $\Phi(y) = (-\Delta)^{-1} y + \phi_0$.

Assumptions on V, A, and $\mathbf{K}$ - Examples

- Consider the following class of compliant obstacle problems where the obstacle is given implicitly by solving a PDE, thus coupling a VI and a PDE:

$$\begin{aligned} y \le \Phi, \quad \langle Ay - f, y - v\rangle &\le 0, & \forall v \in V : v \le \Phi, \\ \langle B\Phi + G(\Phi, y) - g, w\rangle &= 0 & \forall w \in V, \end{aligned}$$

for some G, $B \in \mathcal{L}(V, V')$ and $\langle Bz^-, z^+\rangle \le 0$.

Assumptions on V, A, and $\mathbf{K}$ - Examples

► Consider the following class of compliant obstacle problems where the obstacle is given implicitly by solving a PDE, thus coupling a VI and a PDE:

$$
\begin{aligned}
y \le \Phi, \quad \langle Ay - f, y - v\rangle \le 0, & \qquad \forall v \in V : v \le \Phi, \\
\langle B\Phi + G(\Phi, y) - g, w\rangle = 0 & \qquad \forall w \in V,
\end{aligned}
$$

for some G, $B \in \mathcal{L}(V, V')$ and $\langle Bz^-, z^+\rangle \le 0$.

Example

Consider $a_{ij}, b_{ij}, a_0, b_0 \in L^\infty(\Omega)$ and the elliptic operators

$$
\begin{aligned}
\langle Ay, z\rangle &= \sum_{i,j} \int_\Omega a_{ij}(x) \frac{\partial y}{\partial x_j} \frac{\partial z}{\partial x_i}\, \mathrm{d}x + \int_\Omega a_0(x) y z\, \mathrm{d}x, && \forall y, z \in V, \\
\langle Bv, w\rangle &= \sum_{i,j} \int_\Omega b_{ij}(x) \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_i}\, \mathrm{d}x + \int_\Omega b_0(x) v w\, \mathrm{d}x, && \forall v, w \in V,
\end{aligned}
$$

Additionally, for $y \ge 0$

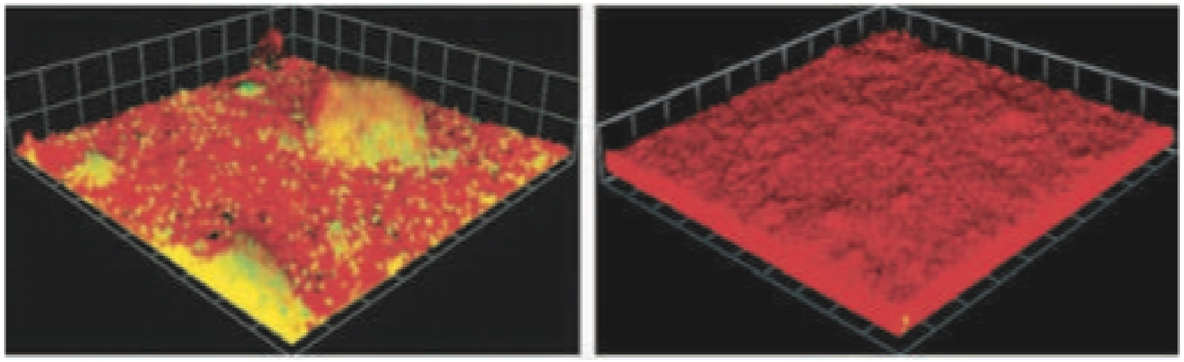
$$
G(\Phi, y) = (\Phi - y)^+.
$$

Some applications

Applications: Competitive Chemotaxis

$(\mathbf{1})$ Let y be the population density (bacteria) and S the nutritional substrate density. If the density is higher than a threshold value and S is sufficiently large, the bacteria bulk (some cases) adheres to that location:

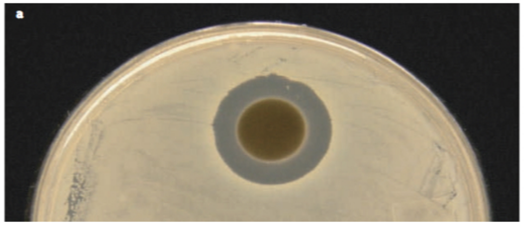
$$y \geq \Phi_1(y, S).$$



Exploitation competition (credit M. E. Hibbing.)

$(\mathbf{2})$ Some bacteria populations generate antimicrobial compounds against competing populations. A bound of the following form arises

$$y_2 \leq \Phi_2(y_1, y_2).$$



Contest competition.

Applications: Thermoforming

Manufacture of products by heating a plastic sheet $u : \Omega \to \mathbb{R}$ and forcing it onto mold $\Phi(u) : \Omega \to \mathbb{R}$

- The contact problem is a VI under perfect sliding of the membrane u with the mould ([Andrä, Warby, Whiteman]).
- Temperature difference between the mold and the plastic sheet $\to$ heat transfer
- Some mold materials change dynamically upon contact $\to$ QVI.

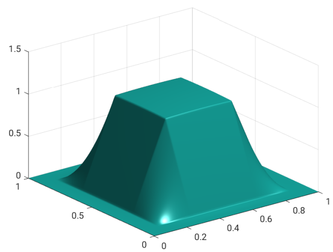
Find $u \in H_0^1(\Omega)$, $u \leq \Phi(u)$,

$$\langle A_1(T)u - f, u - v\rangle \leq 0 \quad \forall v \leq \Phi(u)$$

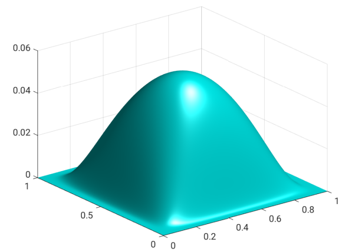
where Φ satisfies as

$$A_2(T) = g(\Phi(u) - u),$$
$$\Phi(u) = \Phi_0 + LT,$$

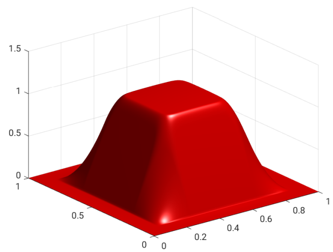
and T is the membrane temperature, $A_2 : H^1(\Omega) \to H^1(\Omega)^*$, and $L \in \mathcal{L}(H^1, H^1)$.



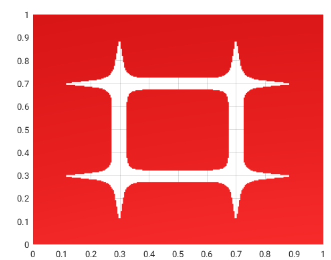
$\Phi(u)$



$\Phi(u) - \Phi_0$



u



$\{u = \Phi(u)\}$

Minimal and maximal solutions

Tartar's Approach

Denote by $S(f, \mathbf{w})$ to the unique solution to

$$\text{Find } y \in \mathbf{K}(\mathbf{w}) : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(\mathbf{w}).$$

The map $S(f, \cdot) : H \to V \subset H$ is well-defined and

- $S(f, \cdot) : H \to H$ is an increasing map:

$$w_0 \leq w_1 \quad \Longrightarrow \quad S(f, w_0) \leq S(f, w_1).$$

- Fixed points of $S(f, \cdot)$ are solutions to the QVI of interest.

Denote by $S(f, \mathbf{w})$ to the unique solution to

$$\text{Find } y \in \mathbf{K}(\mathbf{w}) : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(\mathbf{w}).$$

The map $S(f, \cdot) : H \to V \subset H$ is well-defined and

- $S(f, \cdot) : H \to H$ is an increasing map:

$$w_0 \leq w_1 \quad \Longrightarrow \quad S(f, w_0) \leq S(f, w_1).$$

- Fixed points of $S(f, \cdot)$ are solutions to the QVI of interest.

Theorem (Birkhoff-Tartar)- ([Tartar(1974)])

Let V be a Hilbert space and suppose $T : H \to H$ is an **increasing map**. Let $\underline{y}$ be a **sub-solution** and $\overline{y}$ be a **super-solution** of the map T, that is:

$$\underline{y} \leq T(\underline{y}) \quad \text{and} \quad T(\overline{y}) \leq \overline{y}.$$

If $\underline{y} \leq \overline{y}$, then the set of fixed points of the map T in the interval $[\underline{y}, \overline{y}]$ is non-empty and has a smallest $\mathbf{m}(T)$ and a largest element $\mathbf{M}(T)$.

Definition of $\mathbf{m}$ and $\mathbf{M}$

In general, for applications, sub- and super-solutions of $S(f,\cdot)$ are easy to be found. Let $F \in V'$, and consider that for all admissible forcing terms $f \in U_{ad}$ we have that

$$0 \leq f \leq F.$$

For a given f, we denote by

$$\mathbf{m}(f) \qquad \text{and} \qquad \mathbf{M}(f)$$

to the minimal and maximal solutions of the QVI of interest in the interval $[\underline{y}, \overline{y}] := [0, A^{-1}F]$.

Definition of $\mathbf{m}$ and $\mathbf{M}$

In general, for applications, sub- and super-solutions of $S(f,\cdot)$ are easy to be found. Let $F \in V'$, and consider that for all admissible forcing terms $f \in U_{ad}$ we have that

$$0 \leq f \leq F.$$

For a given f, we denote by

$$\mathbf{m}(f) \qquad \text{and} \qquad \mathbf{M}(f)$$

to the minimal and maximal solutions of the QVI of interest in the interval $[\underline{y}, \overline{y}] := [0, A^{-1}F]$.

- The elements $\mathbf{m}(f)$ and $\mathbf{M}(f)$ are extremal points of $\mathbf{Q}(f)$ on the interval $[\underline{y}, \overline{y}]$:

$$\mathbf{Q}(f) \cap [\underline{y}, \overline{y}] \equiv \mathbf{Q}(f) \cap [\mathbf{m}(f), \mathbf{M}(f)].$$

Definition of $\mathbf{m}$ and $\mathbf{M}$

In general, for applications, sub- and super-solutions of $S(f,\cdot)$ are easy to be found. Let $F \in V'$, and consider that for all admissible forcing terms $f \in U_{ad}$ we have that

$$0 \leq f \leq F.$$

For a given f, we denote by

$$\mathbf{m}(f) \qquad \text{and} \qquad \mathbf{M}(f)$$

to the minimal and maximal solutions of the QVI of interest in the interval $[\underline{y}, \overline{y}] := [0, A^{-1}F]$.

- The elements $\mathbf{m}(f)$ and $\mathbf{M}(f)$ are extremal points of $\mathbf{Q}(f)$ on the interval $[\underline{y}, \overline{y}]$:

$$\mathbf{Q}(f) \cap [\underline{y}, \overline{y}] \equiv \mathbf{Q}(f) \cap [\mathbf{m}(f), \mathbf{M}(f)].$$

Q: How to compute $\mathbf{m}(f)$ and $\mathbf{M}(f)$?

Computing $\mathbf{m}(f)$ and $\mathbf{M}(f)$

- Suppose that admissible forcing terms satisfy $0 \leq f \leq F$ for some $F \in L^\infty(\Omega)$ and $\underline{y} = A^{-1}(0) = 0$ and $\overline{y} = A^{-1}(F)$.
- Let $\Phi : V \to V$ be completely continuous (maps weak into strong)

Computing $\mathbf{m}(f)$ and $\mathbf{M}(f)$

- Suppose that admissible forcing terms satisfy $0 \leq f \leq F$ for some $F \in L^\infty(\Omega)$ and $\underline{y} = A^{-1}(0) = 0$ and $\overline{y} = A^{-1}(F)$.
- Let $\Phi : V \to V$ be completely continuous (maps weak into strong)
- Define the sequences $\{m_n\}$ and $\{M_n\}$ as

$$m_n = S(f, m_{n-1}), \quad n \in \mathbb{N} \qquad\qquad m_0 = \underline{y};$$
$$M_n = S(f, M_{n-1}), \quad n \in \mathbb{N} \qquad\qquad M_0 = \overline{y}.$$

- Then, $m_n \uparrow \mathbf{m}(f)$, $M_n \downarrow \mathbf{M}(f)$,

$$m_n \to \mathbf{m}(f), \text{ and } M_n \to \mathbf{M}(f) \quad \text{in } V$$

  - Convergence (in general) is as slow (sublinear) as you can imagine ; the idea of the proof goes back to Kolodner, Birkhoff, etc....

Open question: Are there simple ways to improve convergence speed?

Perturbations of minimal and maximal solutions

The reduced problem

The problem of interest is

Let $A : V \to V'$ and $f \in V'$ for some (real) Hilbert space V. Consider

$$\text{Find } y \in \mathbf{K}(y) : \langle Ay - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(y) \tag{QVI}$$

where

$$\mathbf{K}(w) := \{z \in V : z \leq \Phi(w)\}.$$

We now require stability results for

$$f \mapsto \mathbf{m}(f) \quad \text{and} \quad f \mapsto \mathbf{M}(f).$$

- What topology on the space of admissible controls?
- What conditions on Φ?

- A. Alphonse, M. Hintermüller, C. N. R.,*Stability of the Solution Set of Quasi-variational Inequalities and Optimal Control*, arXiv:1904.06231, 2019.

Limitations of macro results

Let's recall the Birkhoff-Tartar theorem

Theorem (Birkhoff-Tartar)- ([Tartar(1974)])
Let V be a Hilbert space and suppose $T : H \to H$ is an **increasing map**. Let $\underline{y}$ be a **sub-solution** and $\overline{y}$ be a **super-solution** of the map T, that is:

$$\underline{y} \leq T(\underline{y}) \quad \text{and} \quad T(\overline{y}) \leq \overline{y}.$$

If $\underline{y} \leq \overline{y}$, then the set of fixed points of the map T in the interval $[\underline{y}, \overline{y}]$ is non-empty and has a smallest $\mathbf{m}(T)$ and a largest element $\mathbf{M}(T)$.

- Initially, let's consider (reasonable) approximations of T and try to prove that $\mathbf{m}$ and $\mathbf{M}$ are stable.

Limitations of macro results

- Consider first an increasing map $T : H \to V \subset H$ and that is approximated by maps R_n and U_n from below, and above, respectively.

Proposition. Let $T, R_n, U_n : H \to V \subset H$ be increasing mappings for $n \in \mathbb{N}$ with $T : V \to V$ completely continuous. Suppose that for all $v \in [\underline{y}, \overline{y}]$, and $n \in \mathbb{N}$

$$\underline{y} \leq R_n(v) \leq R_{n+1}(v) \leq T(v) \leq U_{n+1}(v) \leq U_n(v) \leq \overline{y},$$

Limitations of macro results

- Consider first an increasing map $T : H \to V \subset H$ and that is approximated by maps R_n and U_n from below, and above, respectively.

Proposition. Let $T, R_n, U_n : H \to V \subset H$ be increasing mappings for $n \in \mathbb{N}$ with $T : V \to V$ completely continuous. Suppose that for all $v \in [\underline{y}, \overline{y}]$, and $n \in \mathbb{N}$

$$\underline{y} \leq R_n(v) \leq R_{n+1}(v) \leq T(v) \leq U_{n+1}(v) \leq U_n(v) \leq \overline{y},$$

and that if $\{v_n\}$ and $\{w_n\}$ are bounded sequences in V such that $v_n \leq v_{n+1}$ and $w_n \geq w_{n+1}$, then

$$\lim_{n\to\infty} \|R_n(v_n) - T(v_n)\|_V = 0 \qquad \text{and} \quad \lim_{n\to\infty} \|U_n(w_n) - T(w_n)\|_V = 0.$$

Then $\mathbf{m}(R_n) \leq \mathbf{m}(T)$ and $\mathbf{M}(T) \leq \mathbf{M}(U_n)$, and as $n \to \infty$,

$$\mathbf{m}(R_n) \to \mathbf{m}(T) \text{ in } V \qquad \text{and} \qquad \mathbf{M}(U_n) \to \mathbf{M}(T) \text{ in } V.$$

Limitations of macro results

- How tight is the previous result?

Limitations of macro results

- How tight is the previous result?

Let $T : [0,1] \to [0,1]$ be defined as

$$T(v) = \begin{cases} a, & 0 \le v < a; \\ v, & a \le v < b; \\ b, & b \le v \le 1\,. \end{cases}$$

with $0 < a < b < 1$ and where $\mathbf{m}(T) = a$ and $\mathbf{M}(T) = b$ and

$$R_n(v) = \begin{cases} a, & 0 \le v < \frac{1}{n}; \\ T(v - \frac{1}{n}), & \frac{1}{n} \le v \le 1\,. \end{cases} \qquad U_n(v) = \begin{cases} T(v + \frac{1}{n}), & 0 \le v < 1 - \frac{1}{n}; \\ b, & 1 - \frac{1}{n} \le v \le 1\,. \end{cases}$$

Suppose that $n > N$ for N sufficiently large, then all the assumptions of the previous theorem hold, but

$$a = \mathbf{M}(R_n) \not\to \mathbf{M}(T) = b \quad \text{and} \quad b = \mathbf{m}(U_n) \not\to \mathbf{m}(T) = a.$$

Limitations of macro results

- How tight is the previous result?

Let $T : [0,1] \to [0,1]$ be defined as

$$T(v) = \begin{cases} a, & 0 \le v < a; \\ v, & a \le v < b; \\ b, & b \le v \le 1\,. \end{cases}$$

with $0 < a < b < 1$ and where $\mathbf{m}(T) = a$ and $\mathbf{M}(T) = b$ and

$$R_n(v) = \begin{cases} a, & 0 \le v < \frac{1}{n}; \\ T(v - \frac{1}{n}), & \frac{1}{n} \le v \le 1\,. \end{cases} \qquad U_n(v) = \begin{cases} T(v + \frac{1}{n}), & 0 \le v < 1 - \frac{1}{n}; \\ b, & 1 - \frac{1}{n} \le v \le 1\,. \end{cases}$$

Suppose that $n > N$ for N sufficiently large, then all the assumptions of the previous theorem hold, but

$$a = \mathbf{M}(R_n) \not\to \mathbf{M}(T) = b \quad \text{and} \quad b = \mathbf{m}(U_n) \not\to \mathbf{m}(T) = a.$$

Then the proof plan is to consider perturbations $f_n \downarrow f$ and $f_n \uparrow f$ separately.

Lemma 1. Suppose that

i. The sequence $\{f_n\}$ in $L^\infty_\nu(\Omega)$ is non-increasing and converges to f^* in $L^\infty(\Omega)$.

ii. The upper bound mapping Φ satisfies that

$$\lambda\Phi(y) \geq \Phi(\lambda y), \qquad \text{for any} \quad \lambda \geq 1, \;\; y \in V \cap H^+,$$

and if $v_n \to v$ in H, then $\Phi(v_n) \to \Phi(v)$ in H.

Then $\mathbf{m}(f_n) \downarrow \mathbf{m}(f^*)$ in H and

$$\mathbf{m}(f_n) \to \mathbf{m}(f^*) \text{ in } V.$$

- $L^\infty_\nu(\Omega) := \{g \in L^\infty(\Omega) : g \geq \nu > 0\}$.
- Note that we are not assuming that if $v_n \rightharpoonup v$ then Mosco convergence $\mathbf{K}(v_n) \to \mathbf{K}(v)$ holds!

- Q: Why is $L^\infty(\Omega)$ required as the space of perturbations?

- Q: Why is $L^\infty(\Omega)$ required as the space of perturbations?

Then $\mathbf{m}(f)$ is the maximal element of the set $Z^\bullet(f)$ with

$$
\begin{aligned}
X(f) &= \{x \in V : x \in [\underline{y}, \overline{y}] \text{ and } x \le S(f,x)\},\\
Y^\bullet(f) &= \{x \in V : x \in [\underline{y}, \infty) \text{ and } x \ge S(f,x)\},\\
Z^\bullet(f) &= \{x \in X(f) : x \le y \text{ for all } y \in Y^\bullet(f)\}.
\end{aligned}
$$

Similarly, $\mathbf{M}(f)$ is the minimal element of the set $\tilde{Z}^\bullet(f)$ where

$$
\begin{aligned}
X^\bullet(f) &= \{x \in V : x \in (-\infty, \overline{y}] \text{ and } x \le S(f,x)\},\\
Y(f) &= \{x \in V : x \in [\underline{y}, \overline{y}] \text{ and } x \ge S(f,x)\},\\
\tilde{Z}^\bullet(f) &= \{y \in Y(f) : x \le y \text{ for all } x \in X^\bullet(f)\}.
\end{aligned}
$$

- A: The set-valued maps $f \mapsto Z^\bullet(f), \tilde{Z}^\bullet(f)$ are delicate

Non-decreasing Sequences of $\{f_n\}$ and $\mathbf{m}$

Lemma 2. Suppose that

i. The sequence $\{f_n\}$ in V'_+ is non-decreasing and converges to f^* in V'.

ii. The upper bound mapping Φ satisfies one of the following:

 a. If $v_n \rightharpoonup v$ in V, then $\Phi(v_n) \to \Phi(v)$ in $L^\infty(\Omega)$.

 b. If $v_n \rightharpoonup v$ in V, then $\Phi(v_n) \to \Phi(v)$ in H and if $v \in V \cap H^+$, then $\Phi(v) \in V$ and $-\Delta\Phi(v) \geq 0$.

Then $\mathbf{m}(f_n) \uparrow \mathbf{m}(f^*)$ in H and

$$\mathbf{m}(f_n) \to \mathbf{m}(f^*) \text{ in } V.$$

Lemma 3. Suppose that

i. The sequence $\{f_n\}$ in V'_+ is non-increasing and converges to f^* in V'.

ii. The upper bound mapping Φ satisfies: If $v_n \to v$ in H, then $\Phi(v_n) \to \Phi(v)$ in H.

Then $\mathbf{M}(f_n) \downarrow \mathbf{M}(f^*)$ in H and

$$\mathbf{M}(f_n) \to \mathbf{M}(f^*) \text{ in } V.$$

Lemma 4. Suppose that

i. The sequence $\{f_n\}$ in $L^\infty_\nu(\Omega)$ is non-decreasing and converges to f^* in $L^\infty(\Omega)$.

ii. The upper bound mapping Φ satisfies that

$$\lambda\Phi(y) \le \Phi(\lambda y), \quad \text{for any} \quad 0 < \lambda < 1, \quad y \in V \cap H^+,$$

and one of the following:

a. If $v_n \rightharpoonup v$ in V, then $\Phi(v_n) \to \Phi(v)$ in $L^\infty(\Omega)$.

b. If $v_n \rightharpoonup v$ in V, then $\Phi(v_n) \to \Phi(v)$ in H and if $v \in V \cap H^+$, then $\Phi(v) \in V$ and $-\Delta\Phi(v) \ge 0$.

Then $\mathbf{M}(f_n) \uparrow \mathbf{M}(f^*)$ in H and

$$\mathbf{M}(f_n) \to \mathbf{M}(f^*) \text{ in } V.$$

Stability for $\mathbf{M}$ and $\mathbf{m}$

Theorem. Suppose that

i. The sequence $\{f_n\}$ in $L^\infty_\nu(\Omega)$ converges to f^* in $L^\infty(\Omega)$.

ii. The upper bound mapping Φ satisfies the conditions of the previous lemmas. In particular, for any $y \in V \cap H^+$

$$\begin{aligned} \lambda\Phi(y) &\geq \Phi(\lambda y), && \text{for any} && \lambda \geq 1, && \text{or} \\ \lambda\Phi(y) &\leq \Phi(\lambda y), && \text{for any} && 0 < \lambda < 1. \end{aligned}$$

Then

$$\mathbf{m}(f_n) \to \mathbf{m}(f^*) \quad \text{and} \quad \mathbf{M}(f_n) \to \mathbf{M}(f^*) \ \text{ in } \ H,$$

together with

$$\mathbf{m}(f_n) \rightharpoonup \mathbf{m}(f^*) \quad \text{and} \quad \mathbf{M}(f_n) \rightharpoonup \mathbf{M}(f^*) \ \text{ in } \ V.$$

- No order in $\{f_n\}$ $\Rightarrow$ no strong convergence in V.

Example of application

We would like to control the solution set $f \mapsto \mathbf{Q}(f)$ of the QVI

Let $A : V \to V'$ and $f \in V'$ for some (real) Hilbert space V. Consider

$$\text{Find } y \in \mathbf{K}(y) : \langle Ay - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(y) \tag{QVI}$$

where

$$\mathbf{K}(w) := \{z \in V : z \leq \Phi(w)\}.$$

- Suppose that we require that $\mathbf{Q}(f)$ is a singleton:
  - In our setting we would to select a forcing term f such that

$$|\mathbf{m}(f) - \mathbf{M}(f)|_{L^2(\Omega)},$$

is as small as possible in addition to requiring that $\mathbf{m}(f)$ is close to a desired state.

Example of application

Consider the following problem

$$\min_{f\in U_{\text{ad}}} \int_\Omega |\mathbf{m}(f) - \mathbf{M}(f)|^2 + \int_\Omega |\mathbf{m}(f) - y_d|^2,$$

for some admissible control set $U_{\text{ad}} \subset U$ and where $\mathbf{m}(f)$, and $\mathbf{M}(f)$ correspond to the minimal and maximal solutions of the following QVI

Find $y \in \mathbf{K}(y) : \langle A(y) - f, v - y\rangle \geq 0, \quad \forall v \in \mathbf{K}(y).$

Example of application

Consider the following problem

$$\min_{f \in U_{\text{ad}}} \int_\Omega |\mathbf{m}(f) - \mathbf{M}(f)|^2 + \int_\Omega |\mathbf{m}(f) - y_d|^2,$$

for some admissible control set $U_{\text{ad}} \subset U$ and where $\mathbf{m}(f)$, and $\mathbf{M}(f)$ correspond to the minimal and maximal solutions of the following QVI

$$\text{Find } y \in \mathbf{K}(y) : \langle A(y) - f, v - y \rangle \geq 0, \quad \forall v \in \mathbf{K}(y).$$

- If U is compactly embedded in $L^\infty(\Omega)$, U is a reflexive Banach space and $U_{\text{ad}} \subset L^\infty_\nu(\Omega)$ is bounded, then the above problem has a solution (under the assumptions we have described).

- A. Alphonse, M. Hintermüller, C. N. R.,*Stability of the Solution Set of Quasi-variational Inequalities and Optimal Control*, arXiv:1904.06231, 2019.

Directional differentiability of $f \mapsto \mathbf{Q}(f)$.

Directional differentiability

Given $\mathbf{Q}(f)$ the solution set to QVI

We are interested in the directional differentiability of $\mathbf{Q}$: we want to show (formally)

$$\mathbf{Q}(f + td) \supset \mathbf{Q}(f) + t\mathbf{Q}'(f)(d) + o(t)$$

where $t^{-1}o(t) \to 0$ as $t \to 0^+$.

Directional differentiability results useful for

- Optimal control of QVI.
- Numerical methods.

Selected work:

- Sensitivity for VIs and related issues: Alphonse, Bergounioux, Christof, Hintermüller, Haraux, Herzog, Ito, Kunisch, Leugering, Meyer, Mignot, Puel, Surowiec, Sprekels, M. Ulbrich, S. Ulbrich, D. Wachsmuth, G. Wachsmuth, Zarantonello,...

- A. Alphonse, M. Hintermüller, C. N. R.,*Stability of the Solution Set of Quasi-variational Inequalities and Optimal Control*, CoVs and PDEs 58 (1), 39 (2019).

Introduction

We are interested in the directional differentiability of $\mathbf{Q}$: we want to show (formally)

$$\mathbf{Q}(f + td) \supset \mathbf{Q}(f) + t\mathbf{Q}'(f)(d) + o(t)$$

where $t^{-1}o(t) \to 0$ as $t \to 0^+$.

We face the same questions as before

- What topology on the space of admissible controls f and perturbations d?
- What conditions on Φ?
  - For the time being just assume that $\Phi\colon V \to V$ is Hadamard differentiable: That is, for all v and all h in V, the limit

    $$\lim_{\substack{h' \to h \\ t \to 0^+}} \frac{\Phi(v + th') - \Phi(v)}{t}$$

    exists in V, and we write the limit as $\Phi'(v)(h)$.

Main result

Assume $f, d \in L^\infty_+(\Omega)$ and define $\bar{y}, \bar{q}(t)$ by

$$A\bar{y} = f \qquad \& \qquad A\bar{q}(t) = f + td.$$

Main result

Assume $f, d \in L^\infty_+(\Omega)$ and define $\bar{y}, \bar{q}(t)$ by

$$A\bar{y} = f \qquad \& \qquad A\bar{q}(t) = f + td.$$

Existence: by the Tartar–Birkhoff theorem, the following sets are non-empty:

$$\mathbf{Q}(f) \cap [0, \bar{y}] \qquad \& \qquad \mathbf{Q}(f + td) \cap [y, \bar{q}(t)].$$

Main result

Assume $f, d \in L^\infty_+(\Omega)$ and define $\bar{y}, \bar{q}(t)$ by

$$A\bar{y} = f \qquad \& \qquad A\bar{q}(t) = f + td.$$

Existence: by the Tartar–Birkhoff theorem, the following sets are non-empty:

$$\mathbf{Q}(f) \cap [0, \bar{y}] \qquad \& \qquad \mathbf{Q}(f + td) \cap [y, \bar{q}(t)].$$

Theorem. For every $y \in \mathbf{Q}(f) \cap [0, \bar{y}]$, then under certain conditions, there exists $q(t) \in \mathbf{Q}(f + td)$ and $\alpha \in V_+$ such that

$$q(t) = y + t\alpha + o(t).$$

Furthermore, $\alpha = \alpha(d)$ is positively homogeneous and satisfies the QVI

$$\alpha \in \mathcal{K}^y(\alpha) : \langle A\alpha - d, \alpha - v\rangle \leq 0 \quad \forall v \in \mathcal{K}^y(\alpha)$$
$$\mathcal{K}^y(w) := \{\varphi \in V : \varphi \leq \Phi'(y)(w) \text{ q.e. on } \mathcal{A}(y) \;\&\; \langle Ay - f, \varphi - \Phi'(y)(w)\rangle = 0\}.$$

This is an extension of Mignot result for VIs into QVIs

Why didn't we use use Mignot's result (after variable change)?

As before, $S(g,\psi) = y \in V$ be the solution of the VI

$$y \in \mathbf{K}(\psi) : \quad \langle Ay - g, y - v\rangle \le 0 \quad \forall v \in \mathbf{K}(\psi).$$

Mignot 's result implies that $S(\cdot,\psi)$ has a direc. derivative $DS(g,\psi)(d) =: \delta$, i.e.,

$$S(g + td, \psi) = S(g,\psi) + tDS(g,\psi)(d) + o(t,d)$$

where $t^{-1}o(t,d) \to 0$ as $t \to 0^+$ uniformly in d on compact subsets of V^*. It solves

$$\delta \in \mathcal{K}^y : \quad \langle A\delta - d, \delta - v\rangle \le 0 \quad \forall v \in \mathcal{K}^y$$
$$\mathcal{K}^y := \{w \in V : w \le 0 \text{ q.e. on } \{y = \Phi(\psi)\} \text{ and } \langle Ay - g, w\rangle = 0\}.$$

Why didn't we use use Mignot's result (after variable change)?

As before, $S(g,\psi) = y \in V$ be the solution of the VI

$$y \in \mathbf{K}(\psi): \quad \langle Ay - g, y - v\rangle \le 0 \quad \forall v \in \mathbf{K}(\psi).$$

Mignot 's result implies that $S(\cdot,\psi)$ has a direc. derivative $DS(g,\psi)(d) =: \delta$, i.e.,

$$S(g + td, \psi) = S(g,\psi) + tDS(g,\psi)(d) + o(t,d)$$

where $t^{-1}o(t,d) \to 0$ as $t \to 0^+$ uniformly in d on compact subsets of V^*. It solves

$$\delta \in \mathcal{K}^y: \quad \langle A\delta - d, \delta - v\rangle \le 0 \quad \forall v \in \mathcal{K}^y$$
$$\mathcal{K}^y := \{w \in V : w \le 0 \text{ q.e. on } \{y = \Phi(\psi)\} \text{ and } \langle Ay - g, w\rangle = 0\}.$$

Transforming $y = \Phi(y) - \hat{y}$ leads to

$$\hat{y} \in \mathbf{K}_0: \quad \langle \hat{A}\hat{y} - \hat{f}, \hat{y} - \varphi\rangle \le 0 \quad \forall \varphi \in \mathbf{K}_0$$

where $\hat{A} := A - A\Phi(\Phi - I)^{-1}$ and $\hat{f} = -f$.

In general, $\hat{A}$ is not linear, nor coercive, nor T-monotone $\to$ New math needed.

Proof plan

In the formal equality

$$\mathbf{Q}(f + td) \supset \mathbf{Q}(f) + t\mathbf{Q}'(f)(d) + o(t), \tag{1}$$

1. Select an element $y \in \mathbf{Q}(f)$.
2. Approximate an associated element $q(t)$ to y from $\mathbf{Q}(f + td)$ by a sequence $q_n(t)$ of solutions of VIs

Proof plan

In the formal equality

$$\mathbf{Q}(f + td) \supset \mathbf{Q}(f) + t\mathbf{Q}'(f)(d) + o(t), \tag{1}$$

1. Select an element $y \in \mathbf{Q}(f)$.
2. Approximate an associated element $q(t)$ to y from $\mathbf{Q}(f + td)$ by a sequence $q_n(t)$ of solutions of VIs
3. Obtain differential formulae for the q_n and characterize the approximation α_n to the directional derivative.
   (must relate q_n to y; recursion plays a highly nonlinear role)

Proof plan

In the formal equality

$$\mathbf{Q}(f+td) \supset \mathbf{Q}(f) + t\mathbf{Q}'(f)(d) + o(t), \tag{1}$$

1. Select an element $y \in \mathbf{Q}(f)$.
2. Approximate an associated element $q(t)$ to y from $\mathbf{Q}(f+td)$ by a sequence $q_n(t)$ of solutions of VIs
3. Obtain differential formulae for the q_n and characterize the approximation α_n to the directional derivative.
   (must relate q_n to y; recursion plays a highly nonlinear role)
4. Pass to the limit to hopefully retrieve (1)
   (handling a recurrence inequality to obtain uniform bounds + identifying the limit of higher order terms as a higher order term).

Proof Plan (1/2) - “Sequential expressions”

As usual, $S(g,\psi) = z \in V$ is the solution to the VI

$$z \in \mathbf{K}(\psi) : \quad \langle Az - g, z - v \rangle \le 0 \quad \forall v \in \mathbf{K}(\psi).$$

Proof Plan (1/2) - “Sequential expressions”

As usual, $S(g, \psi) = z \in V$ is the solution to the VI

$$z \in \mathbf{K}(\psi) : \quad \langle Az - g, z - v \rangle \leq 0 \quad \forall v \in \mathbf{K}(\psi).$$

Define $q_n := S(f + td, q_{n-1})$ and $q_0 := y$.

Proof Plan (1/2) - “Sequential expressions”

As usual, $S(g,\psi) = z \in V$ is the solution to the VI

$$z \in \mathbf{K}(\psi) : \quad \langle Az - g, z - v\rangle \leq 0 \quad \forall v \in \mathbf{K}(\psi).$$

Define $q_n := S(f + td, q_{n-1})$ and $q_0 := y$.

Lemma. For each n,

$$q_n(t) = y + t\alpha_n + o_n(t)$$

where

$$\begin{aligned} \alpha_n &= \Phi'(y)[\alpha_{n-1}] + DS(f,y)[d - A\Phi'(y)(\alpha_{n-1})] \\ o_n(t) &= r(t, \alpha_{n-1}, o_{n-1}(t)) \end{aligned}$$

Proof Plan (1/2) - "Sequential expressions"

As usual, $S(g, \psi) = z \in V$ is the solution to the VI

$$z \in \mathbf{K}(\psi) : \quad \langle Az - g, z - v \rangle \leq 0 \quad \forall v \in \mathbf{K}(\psi).$$

Define $q_n := S(f + td, q_{n-1})$ and $q_0 := y$.

Lemma. For each n,

$$q_n(t) = y + t\alpha_n + o_n(t)$$

where

$$\begin{aligned} \alpha_n &= \Phi'(y)[\alpha_{n-1}] + DS(f, y)[d - A\Phi'(y)(\alpha_{n-1})] \\ o_n(t) &= r(t, \alpha_{n-1}, o_{n-1}(t)) \end{aligned}$$

and $t^{-1}o_n(t) \to 0$ as $t \to 0^+$ and α_n is positively homogeneous in d and solves the VI

$$\alpha_n \in \mathcal{K}^y(\alpha_{n-1}) : \langle A\alpha_n - d, \alpha_n - \varphi \rangle \leq 0 \qquad \forall \varphi \in \mathcal{K}^y(\alpha_{n-1})$$

$$\mathcal{K}^y(\alpha_{n-1}) = \{\varphi \in V : \varphi \leq \Phi'(y)(\alpha_{n-1}) \text{ q.e. on } \mathcal{A}(y)$$

$$\& \ \langle Ay - f, \varphi - \Phi'(y)(\alpha_{n-1}) \rangle = 0.\}$$

Proof Plan (2/2)- “Little o asymptotics”

Let c and C be the coercivity and boundedness constants of A, respectively.

Lemma. If there exists $\epsilon > 0$ such that $\|\Phi'(y)b\|_V \leq (c-\epsilon)/C\|b\|_V$, then α_n is bounded in V.

Proof Plan (2/2)- “Little o asymptotics”

Let c and C be the coercivity and boundedness constants of A, respectively.

Lemma. If there exists $\epsilon > 0$ such that $\|\Phi'(y)b\|_V \leq (c - \epsilon)/C\|b\|_V$, then α_n is bounded in V.

Since $o_n(t) = q_n(t) - u - t\alpha_n$,

$$o_n(t) \rightharpoonup o^*(t) \text{ in } V.$$

Proof Plan (2/2)- "Little o asymptotics"

Let c and C be the coercivity and boundedness constants of A, respectively.

Lemma. If there exists $\epsilon > 0$ such that $\|\Phi'(y)b\|_V \leq (c-\epsilon)/C\|b\|_V$, then α_n is bounded in V.

Since $o_n(t) = q_n(t) - u - t\alpha_n$,

$$o_n(t) \rightharpoonup o^*(t) \text{ in } V.$$

Is o^* a higher order term?

Proof Plan (2/2)- “Little o asymptotics”

Let c and C be the coercivity and boundedness constants of A, respectively.

Lemma. If there exists $\epsilon > 0$ such that $\|\Phi'(y)b\|_V \leq (c-\epsilon)/C\|b\|_V$, then α_n is bounded in V.

Since $o_n(t) = q_n(t) - u - t\alpha_n$,

$$o_n(t) \rightharpoonup o^*(t) \text{ in } V.$$

Is o^* a higher order term?

Assume
(1) $V \ni z \to \Phi'(v)(z) \in V$ is completely continuous,
(2) for $T_0 \in (0,T)$ small, if $z\colon (0,T_0) \to V$ satisfies $z(t) \to y$ as $t \to 0^+$, then

$$\|\Phi'(z(t))b\|_V \leq C_\Phi \|b\|_V \quad \text{where } C_\Phi < (1 + c^{-1}C)^{-1}.$$

Lemma. The convergence $t^{-1}o_n(t) \to 0$ in V as $t \to 0^+$ is uniform in n.

What about $f \mapsto \mathbf{m}(f), \mathbf{M}(f)$?

Can we differentiate the minimal and maximal solutions maps with the previous result?

What about $f \mapsto \mathbf{m}(f), \mathbf{M}(f)$?

Can we differentiate the minimal and maximal solutions maps with the previous result? **Yes and No**:

- For the map $\mathbf{m}$ we have

$$\mathbf{m}(f + td) = \mathbf{m}(f) + t\mathbf{m}'(f)(d) + o(t)$$

where $t^{-1}o(t) \to 0$ as $t \to 0^+$.

- For the map $\mathbf{M}$ we have

$$q(t) = \mathbf{M}(f) + t\alpha(d) + o(t)$$

for some $q(t) \in \mathbf{Q}(f + td)$.

We need to “reverse” orders in the proof to obtain the result for $\mathbf{M}$.

A Thermoforming Model

Aim: manufacture products by heating membrane/sheet and forcing it onto mould

Modelling assumptions (for a time step in the semi-discretisation of the problem):

1. Temperature for the membrane is constant. Position denoted by y
2. Φ grows in an affine fashion w. r. t. its temperature. Position denoted by $\Phi(y)$
3. Temperature T of the mould is subject to diffusion + insulated BCs + vertical distance to membrane.

We consider $V = H_0^1(\Omega)$

$$y \in V : y \leq \Phi(y), \quad \langle Ay - f, y - v\rangle \leq 0 \quad \forall v \in V : v \leq \Phi(y)$$

$$-\Delta T + T = g(\Phi(y) - y) \qquad \text{on } \Omega$$

$$\Phi(u) = \Phi_0 + LT \qquad \text{on } \Omega,$$

where L is a bounded linear increasing operator.

A Thermoforming Model

Aim: manufacture products by heating membrane/sheet and forcing it onto mould

Modelling assumptions (for a time step in the semi-discretisation of the problem):

1. Temperature for the membrane is constant. Position denoted by y
2. Φ grows in an affine fashion w. r. t. its temperature. Position denoted by $\Phi(y)$
3. Temperature T of the mould is subject to diffusion + insulated BCs + vertical distance to membrane.

We consider $V = H_0^1(\Omega)$

$$y \in V : y \leq \Phi(y), \quad \langle Ay - f, y - v \rangle \leq 0 \quad \forall v \in V : v \leq \Phi(y)$$

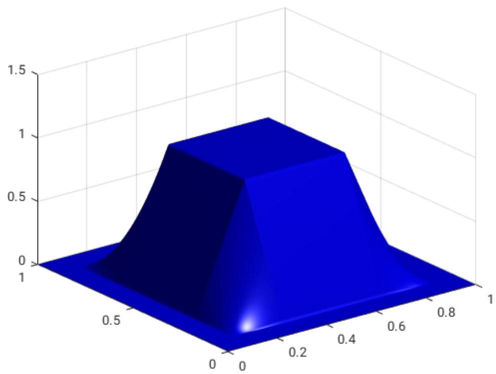
$$-\Delta T + T = g(\Phi(y) - y) \qquad \text{on } \Omega$$

$$\Phi(u) = \Phi_0 + LT \qquad \text{on } \Omega,$$

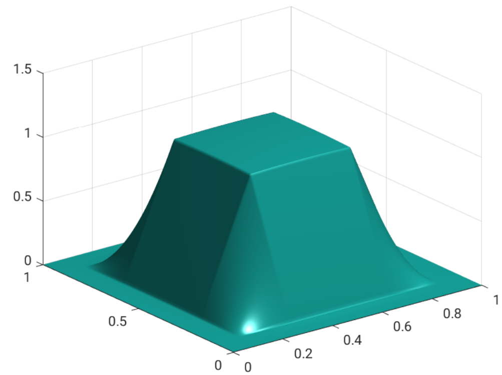
where L is a bounded linear increasing operator.

Provided that g is regular enough, the previous result can be applied and a directional derivative exists.

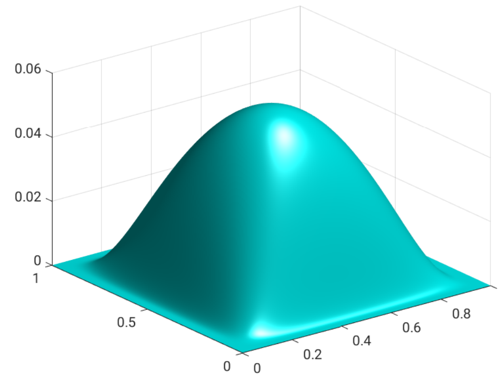
Initial mould Φ_0



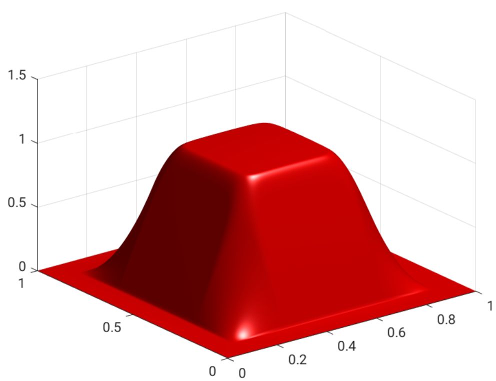
Final mould $\Phi(u)$



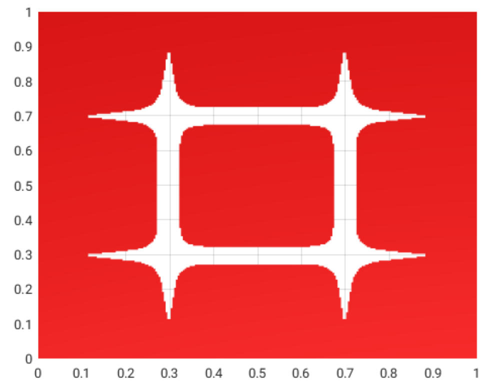
Difference b/w moulds



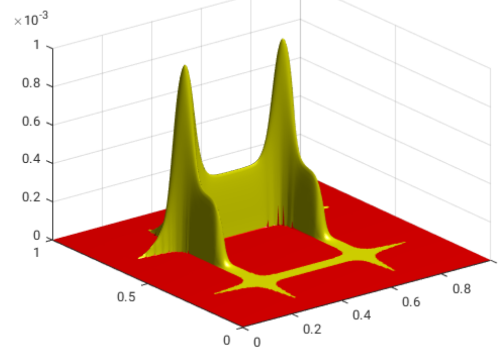
Membrane u for f constant



Active set $\{u = \Phi(u)\}$



Dir. deriv. for $d = \chi_{\{x_1 > 0.5\}}$



Thanks for your attention!