SOLUTIONS AND APPROXIMATIONS TO THE RICCATI INTEGRAL EQUATION WITH VALUES IN A SPACE OF COMPACT OPERATORS*

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Abstract. In this paper, we present conditions that ensure the existence of Bochner integrable solutions of infinite dimensional Riccati integral equations. In particular, we focus on $\mathscr{I}_p$-valued continuous solutions. We formulate an optimal sensor location problem that is based on optimal filtering and show that when the underlying system is of convection-diffusion type the Riccati integral equation has Bochner integrable solutions. We use these results to approximate the sensor placement problem by using a simple quadrature rule. A numerical example is given to illustrate the results.

Key words. Riccati Equations, Bochner Integral, Optimal Sensor Location

AMS subject classifications. 47J05, 60G35

1. Introduction and Notation.

The infinite dimensional Riccati equation is a fundamental topic of optimal control and optimal state estimation and filtering (see, for example [22] and [20] and the references therein). It is known that in some theoretical frameworks (see [20] and [8, 10], for example) it is required that the solution to the Riccati equation is trace class-valued. A comprehensive study for trace class solutions was taken by Bensoussan in [8] and some applications of earlier results can be traced back to [7]. Some fundamental results in Hilbert spaces on problems arising on optimal control were given by Gibson in [36]. The approximation and existence of solutions considering Hilbert-Schmidt valued solutions was considered in [24] and [35], and further fundamental results concerning approximation procedures can be found in [49], [41] and [56].

A main goal of this paper is to establish conditions that imply the existence of Bochner integrable solutions of infinite dimensional Riccati integral equations with values in the Schatten p-classes. This work is motivated by the observation that if the Riccati integral equation can be interpreted as a Bochner integral equation, then simple numerical quadratures can be employed to approximate the equations.

1.1. A Motivating Example and Challenges.

In order to motivate the theoretical developments included here we consider the following convection-diffusion process in the $n-$dimensional unit cube $\Omega = (0,1)^n \subset \mathbb{R}^n$ given by

$$\frac{\partial}{\partial t}T = (c^2\Delta + \mathbf{a}(x)\cdot\nabla)T + b(t,x)\eta(t), \tag{1.1}$$

where $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ is the Laplacian, $\mathbf{a}(x)\cdot\nabla = \sum_{i=1}^n a_i(x)\partial/\partial x_i$ is the convection operator and the maps $x \mapsto a_i(x)$ are regular for $x \in \overline{\Omega}$. We assume η is a real-valued Wiener process (a zero mean Gaussian process) and $b(t,\cdot) \in L^2(\Omega)$ for

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each $t \in [0, t_f]$. The map $b(\cdot, \cdot)$ determines different noise intensities in different regions of the domain and at different times. The boundary and initial conditions are determined by

$$T(t,x)\Big|_{\partial\Omega} = 0, \qquad\qquad T(0,x) = T_0(x) + \xi,$$

where $T_0(\cdot) \in L^2(\Omega)$ and ξ is a $L^2(\Omega)$-valued gaussian random variable. The differential operator $A := (c^2\Delta + \mathbf{a}(x) \cdot \nabla)$, for $c > 0$, has a domain $\mathscr{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and the state space for the problem is $L^2(\Omega)$ since the boundary $\partial\Omega$ is Lipschitz.

We assume that measurements are carried out through p sensor-platforms in Ω, each with a sensor that measures an average value of $T(t, x)$ within an effective range from the location of the platform. Let $\bar{x}_i \in \Omega$, $i = 1, 2, ..., p$ denote the position of the i^{th} sensor and let h_i denote the measured output which is given by

$$h_i(t) = \int_\Omega K(t, x, \bar{x}_i) T(t, x) \ \mathrm{d}x + \nu_i(t) \ . \tag{1.2}$$

Here, the kernel K is a weight and each $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ is a zero-mean *white* noise process and is uncorrelated with η in (1.1). The time dependence of the map K allow us to take into account effects like degradation of the sensor quality and other time dependent effects. This setting includes the outputs considered by Khapalov (see [42], [43], [44], [45] and [46]) and provides a structure that allows for a mathematically rigorous analysis.

A sensor network $\bar{x}_i \in \Omega$, $i = 1, 2, ..., p$, induces the output map $C(t) : L_2(\Omega) \to \mathbb{R}^p$ given by

$$C(t)\varphi = (C_1(t)\varphi,\ C_2(t)\varphi,\ C_3(t)\varphi, \ldots,\ C_p(t)\varphi)^T \in \mathbb{R}^p, \tag{1.3}$$

with each C_i defined as

$$C_i(t)\varphi := \int_\Omega K(t, x, \bar{x}_i)\varphi(x) \ \mathrm{d}x. \tag{1.4}$$

Based on the previous development, we can formulate the abstract (infinite dimensional) model

$$\dot{z}(t) = Az(t) + B(t)\eta(t), \tag{1.5}$$
$$h(t) = C(t)z(t) + \nu(t), \tag{1.6}$$

where $z(0) = z_0 + \xi$ and the state is $z(t)(\cdot) = T(t, \cdot) \in L_2(\Omega)$. In general, we assume that A is the infinitesimal generator of a C_0-semigroup of operators $S(t)$ over $L^2(\Omega)$. This standard abstract formulation allows the extension to include the case where η is an X-valued Wiener process for some separable Hilbert space X and $B(t) \in \mathscr{L}(X, L^2(\Omega))$ for each $t \in [0, t_f]$.

In order to provide criteria for optimal estimation we observe that the variance equation for the optimal estimator is the (weak) solution to an infinite dimensional Riccati (partial) differential equation of the form

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A^* + BR_2B^*(t) - \Sigma(t)(C^*R_1^{-1}C)(t)\Sigma(t) \ , \tag{1.7}$$

with initial condition $\Sigma(0) = \Sigma_0$ which, under certain regularity conditions on the maps $B(\cdot)$ and $C(\cdot)$, it can be proven (see the remark at the end of section 3.3) is also the solution to

$$\Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)(BR_2B^* - \Sigma(s)(C^*R_1^{-1}C)(s)\Sigma(s))S^*(t-s)\,\mathrm{d}s.$$

The operators $R_1(\cdot)$ and $R_2(\cdot)$ are the *incremental covariances* of the uncorrelated Wiener processes η and ν, respectively, and Σ_0 is the covariance operator of the $L^2(\Omega)$-valued Gaussian random variable ξ (see [20] and [7]). If $\hat{z}(\cdot)$ is the stochastic $L^2(\Omega)$-valued process solution to the generalized Kalman-Bucy filter (see [7] and [8, 10]), then the expected value of $\|z(t) - \hat{z}(t)\|^2$ is the trace of the solution to the infinite dimensional Riccati equation at time t, i.e.,

$$\mathbb{E}\big\{\|z(t) - \hat{z}(t)\|^2\big\} = \mathrm{Tr}\,\Sigma(t).$$

It follows that for a sensor network defined by $\{\bar{x}_i\}_{i=1}^p$, the trace of the solution to the Riccati equation is an indicator of the error between the state and the state estimator. In particular, this can be used to define the optimal sensor location problem: Proceeding as in [50], we consider the distributed parameter optimal control problem of finding the locations $\{\bar{x}_i\}_{i=1}^p$ to minimize

$$J(\bar{x}_1, \bar{x}_1, \cdots, \bar{x}_p) = \int_0^{t_f} \mathrm{Tr}\; Q(t)\; \Sigma(t)\; \mathrm{d}t \tag{1.8}$$

where $\Sigma(\cdot)$ is the mild solution of (1.7) and for each $t \in [0, t_f]$, the operator $Q(t) : L_2(\Omega) \to L_2(\Omega)$ is a bounded linear operator. The (time-varying) map $Q(\cdot)$ allows one to weigh significant parts of the state estimate.

Several technical and computational challenges must be addressed in order prove that (1.8) is well-posed. In general, all obstacles in this problem involve the study of solutions $\Sigma(\cdot)$ of the Riccati equation (its regularity, range space, etc), its stability with respect to perturbations on A, B, C and approximation schemes for solutions. Note that, the variance equation is posed in infinite dimensions, then it must be proven that the map $Q(\cdot)\Sigma(\cdot)$ is point-wise of trace class and integrable so that the cost functional in (1.8) is well-defined. Provided that $Q(t) \in \mathscr{I}_q$ (the Schatten q-class, see Definition 1.3) , this implies to obtain that $\Sigma(t) \in \mathscr{I}_p$, where $1/q + 1/p = 1$, for all t. This can be a nontrivial problem, but foundational results in [22], [24], [35], [36], [49], and [56] provide a background towards obtaining solutions in Schatten p-classes. The solution to the problem requires the introduction of approximations and numerical algorithms with appropriate convergence. The basic theory and approximation schemes developed in [14], [16], [24], [35], [36], [41], [49], and [56] determine starting points to overcome this obstacle.

We provide in this paper a general theoretical framework to deal with solutions of the Riccati equation not only of trace class, but on all Schatten p-classes. We consider an approach based on approximations that can be used not only for theoretical purposes but also for approximation schemes that are suitable of numerical implemention.

1.2. Notation and preliminaries. Let $\mathscr{H}$ be a separable complex Hilbert space. The space of bounded linear operators from $\mathscr{H}$ to $\mathscr{H}$ is denoted by $\mathscr{L}(\mathscr{H})$. If $A \in \mathscr{L}(\mathscr{H})$, then $\|A\|$ denotes the usual operator norm. The subspace of compact bounded linear operators acting on $\mathscr{H}$ is denoted by $\mathscr{I}_\infty(\mathscr{H})$ and, when $\mathscr{H}$ is understood, we simply use $\mathscr{I}_\infty$ for $\mathscr{I}_\infty(\mathscr{H})$. It is well known (see for example [53]) that $\mathscr{I}_\infty$ is a two-sided $*$-ideal in the ring $\mathscr{L}(\mathscr{H})$, i.e., $\mathscr{I}_\infty$ is a vector space and;

1) If $A \in \mathscr{I}_\infty$ and $B \in \mathscr{L}(\mathscr{H})$, then $AB \in \mathscr{I}_\infty$ and $BA \in \mathscr{I}_\infty$.
2) If $A \in \mathscr{I}_\infty$ then $A^* \in \mathscr{I}_\infty$.

Also, finite rank operators are dense (in the operator norm) in $\mathscr{I}_\infty$ and if $A_n \in \mathscr{I}_\infty$ for each $n \in \mathbb{N}$ and $\|A_n - A\| \to 0$ as $n \to \infty$, then $A \in \mathscr{I}_\infty$.

DEFINITION 1.1. *An operator $A \in \mathscr{L}(\mathscr{H})$, is said to be* non-negative *if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$,* positive *if $\langle Ax, x\rangle > 0$ for all nonzero $x \in \mathscr{H}$ and* strictly positive *if there is a $c > 0$ such $\langle Ax, x\rangle \geq c\|x\|^2$ for all $x \in \mathscr{H}$.*

It should be noted that since $\mathscr{H}$ is a complex Hilbert space a non-negative (positive, or strictly positive) operator is self-adjoint (see VI.4 in [53]).

The notation $A \geq 0$, $A > 0$ and $A \gg 0$ is standard for non-negative, positive and strictly positive operators, respectively. Suppose that $A \geq 0$, and that both $\{\phi_n\}$ and $\{\psi_n\}$ are orthonormal bases of $\mathscr{H}$, then it follows that $\sum_n \langle \phi_n, A\phi_n\rangle = \sum_n \langle \psi_n, A\psi_n\rangle$ (we allow the case where both quantities are infinite). This observation motivates the definition of trace of an operator.

DEFINITION 1.2. *If $A \geq 0$, then the trace of A is defined by*

$$\mathrm{Tr}\,(A) := \sum_{n=1}^{\infty} \langle \phi_n, A\phi_n\rangle,$$

where $\{\phi_n\}_{n=1}^{\infty}$ is any orthonormal basis of $\mathscr{H}$.

Each operator $A \in \mathscr{L}(\mathscr{H})$ admits a *polar decomposition* (see for example VI.4 in [53] or 3.9 in [19]) analogous to the decomposition $z = e^{i\mathrm{Arg}(z)}|z|$ when $z \in \mathbb{C}$. In particular, let $|A|$ be defined to be the unique non-negative operator (and hence self-adjoint) such that $A = U|A|$, where U is the unique partial isometry such Ker U = Ker $|A|$. It can be shown that $|A| = \sqrt{A^*A}$ (some authors define $|A|$ like this) which is proven to be well-defined by the continuous functional calculus since $A^*A \geq 0$. Furthermore, a constructive sequential monotone approach can be considered to obtain $\sqrt{A^*A}$ (see for example chapter VII in [54]) . Since $|A| \geq 0$, then $|A|^p \geq 0$ for any $p \in \mathbb{N}$ and applying standard continuous functional calculus we can prove that $|A|^p \geq 0$ for any $1 \leq p < \infty$. Hence the quantity $\mathrm{Tr}\,(|A|^p)$ is well defined and leads to the following definition.

DEFINITION 1.3. *Let $\mathscr{I}_p(\mathscr{H})$ for $1 \leq p < \infty$ (or simply $\mathscr{I}_p$ when the space $\mathscr{H}$ is understood) denote the set of all bounded operators over $\mathscr{H}$ such that* $\mathrm{Tr}\,(|A|^p) < \infty$. *If $A \in \mathscr{I}_p(\mathscr{H})$, then the $\mathscr{I}_p$-norm (or just the p-norm) of A is defined as $\|A\|_p := \big(\mathrm{Tr}\,(|A|^p)\big)^{1/p} < \infty$.*

If $\mathscr{H}$ is a complex separable Hilbert space, then the linear space $\mathscr{I}_p$, endowed with the p-norm is a Banach space (see [55]). We focus on the spaces $\mathscr{I}_1$ and $\mathscr{I}_2$ in order to develop a proper framework in which to study the solutions of the Riccati equation. The classes $\mathscr{I}_1$ and $\mathscr{I}_2$ are called the space of Trace Class (or Nuclear) operators and the space of Hilbert-Schmidt operators, respectively. Actually, the space $\mathscr{I}_2$ is a Hilbert space under the inner product

$$\langle A, B\rangle_{\mathscr{I}_2} = \sum_{n=1}^{\infty} \langle A\phi_n, B\phi_n\rangle_{\mathscr{H}},$$

where $A, B \in \mathscr{I}_2$ and $\{\phi_n\}_{n=1}^{\infty}$ is any orthonormal basis of $\mathscr{H}$. Note that $\langle A, A\rangle_{\mathscr{I}_2} = \sum_{n=1}^{\infty} \langle \phi_n, A^*A\phi_n\rangle_{\mathscr{H}}$. The operator $|A|$ is given by $|A| = \sqrt{A^*A}$, and the continuous functional calculus implies that $|A|^2 = (\sqrt{A^*A})^2 = A^*A$. Consequently $\langle A, A\rangle_{\mathscr{I}_2} = \mathrm{Tr}\,(|A|^2) = \|A\|_2^2$.

It is also well known that $\mathscr{I}_p$ is a two-sided $*$-ideal in the ring $\mathscr{L}(\mathscr{H})$ (see [37] for a proof) and that if $1 \leq p_1 < p_2 \leq \infty$, and $A \in \mathscr{I}_{p_1}$ then $A \in \mathscr{I}_{p_2}$ and $\|A\|_{p_2} \leq$

$\|A\|_{p_1}$. Therefore, we have the continuous embedding: $\mathscr{I}_{p_1} \hookrightarrow \mathscr{I}_{p_2}$. As a result of this embedding, it follows by setting $p_2 = \infty$, that every operator in $\mathscr{I}_p$ is compact (See [30], [37] or [55]) and that $\|A\| \le \|A\|_p$ for all $1 \le p \le \infty$. We shall also need the following results (see [30], [37] and/or [55] for proof).

LEMMA 1.4. *If $A \in \mathscr{I}_p$ with $1 \le p \le \infty$ and $B \in \mathscr{I}_q$ where $1/p + 1/q = 1$, then $AB, BA \in \mathscr{I}_1$ and*

$$\|AB\|_1 \le \|A\|_p \|B\|_q, \qquad \|BA\|_1 \le \|A\|_p \|B\|_q. \tag{1.9}$$

Moreover, $\|A\|_p = \|A^*\|_p$ and for any positive integer r we have $A^r \in \mathscr{I}_{p/r}$ and $\|A^r\|_{p/r} \le (\|A\|_p)^r$.

The trace is a continuous linear functional over $\mathscr{I}_1$ (see [30]). Consequently, if $A \in \mathscr{I}_1$, the value $\mathrm{Tr}\,(A) = \sum_{n=1}^{\infty} \langle \phi_n, A\phi_n \rangle$ does not depend on the choice of the orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$. This result, combined with the previous Lemma, gives a simple characterization to the dual spaces of $\mathscr{I}_p$ (see [37]) given by the following Proposition.

PROPOSITION 1.5. *Let φ be a continuous linear functional over $\mathscr{I}_p$ with $1 < p \le \infty$, then there is an operator $A \in \mathscr{I}_q$ with $1/p + 1/q = 1$ such that $\varphi(X) = \mathrm{Tr}\,(AX)$, for all $X \in \mathscr{I}_p$, and $\|\varphi\|_{\mathscr{L}(X,\mathbb{C})} = \|A\|_q$. If φ is a bounded linear functional on $\mathscr{I}_1$, then there is a bounded linear operator $A \in \mathscr{L}(\mathscr{H})$ such that $\varphi(X) = \mathrm{Tr}\,(AX)$ for all $X \in \mathscr{I}_1$ and $\|\varphi\|_{\mathscr{L}(X,\mathbb{C})} = \|A\|$.*

The previous proposition implies that $(\mathscr{I}_p)^* \simeq \mathscr{I}_q$ when when $1 < p \le \infty$ and then $\mathscr{I}_p$ is reflexive when $1 < p < \infty$. Moreover $(\mathscr{I}_1)^* \simeq \mathscr{L}(\mathscr{H})$. If $A \in \mathscr{I}_\infty$, then it is well known (see [37]) that it has a norm convergent expansion given by

$$A(\cdot) = \sum_{n=1}^{\omega} s_n(A) \langle \phi_n, \cdot \rangle \psi_n,$$

with ω possibly infinite and $\{\phi_n\}_{n=1}^{\omega}$ and $\{\psi_n\}_{n=1}^{\omega}$ orthonormal sequences in $\mathscr{H}$. The elements of the sequence $\{s_n(A)\}_{n=1}^{\omega}$ are uniquely determined and called the *singular values of* A. In addition the singular values satisfy $s_n(A) \ge 0$ and $s_1(A) \ge s_2(A) \ge \cdots \ge 0$.

There are several equivalent ways to define the norm $\|A\|_p$ for an $A \in \mathscr{I}_p$. The following result uses the singular values of A and the results of the dual space of $\mathscr{I}_p$ to characterize $\|A\|_p$ (see [30] and [55]).

PROPOSITION 1.6. *Let $A \in \mathscr{I}_p$ and $\{s_j(A)\}_{j=1}^{\omega}$ be its singular values and denote by $\mathscr{I}^0$ to the set of nonzero finite rank operators. Then, if $\frac{1}{p} + \frac{1}{q} = 1$, the norm $\|A\|_p$ satisfies*

$$\|A\|_p = \sup_{W \in \mathscr{I}^0} \frac{|\mathrm{Tr}\,(WA)\,|}{\|W\|_q} = \Big(\sum_{j=1}^{\omega} s_j^p(A) \Big)^{1/p}. \tag{1.10}$$

Suppose that I is a real interval (bounded or unbounded) and that X is a Banach space. We define the space $\mathscr{C}(I;X)$ by

$$\mathscr{C}(I;X) = \Big\{ F : I \to X : t \mapsto F(t) \text{ is continuous in } \|\cdot\|_X \Big\}.$$

If I is closed, then $\mathscr{C}(I;X)$ is a Banach space under the usual sup norm; $\|F(\cdot)\|_{\mathscr{C}(I;X)} = \sup_{t\in I} \|F(t)\|_X$. If $1 \le p_1 < p_2 \le \infty$, then the continuous embedding $\mathscr{I}_{p_1} \hookrightarrow \mathscr{I}_{p_2}$, implies that $\mathscr{C}(I;\mathscr{I}_{p_1}) \hookrightarrow \mathscr{C}(I;\mathscr{I}_{p_2})$ and since $\mathscr{I}_p \hookrightarrow \mathscr{L}(\mathscr{H})$ for any $1 \le p \le \infty$, it follows that $\mathscr{C}(I;\mathscr{I}_p) \hookrightarrow \mathscr{C}(I;\mathscr{L}(\mathscr{H}))$.

1.3. Properties of $\mathscr{I}_p$-valued Mappings. We are interested in mappings of the form $f : I \to X$, where I is an interval (bounded or unbounded) in $\mathbb{R}$ and X is a complex Banach space, and more specifically a space of operators over some Hilbert space $\mathscr{H}$. For this matter, we make use of the Bochner integral, its associated concept of measurability and we refer the reader to [1] and [39] for further definitions and elementary properties of the Bochner integral.

A function $f : I \to X$ is called *simple* if it is of the form $f(t) = \sum_{r=1}^{n} x_r \chi_{\Delta_r}(t)$ for some $n \in \mathbb{N}$, $x_r \in X$ and Lebesgue measurable sets Δ_r. Here, the measure of Δ_r is denoted by $m(\Delta_r)$ and χ_{Δ_r} is the characteristic function of the set Δ_r. The function f is called a step function if each Δ_r can be chosen to be an interval.

Recall that a function is called Bochner measurable (or simply measurable) if there is a sequence of simple functions $f_n : I \to X$ such that $f(t) = \lim_{n\to\infty} f_n(t)$, a.e. for $t \in I$. If X is an operator space, for example $X = \mathscr{L}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, then we can define a weaker form of measurability. We say that a map $T : I \to \mathscr{L}(\mathscr{H})$ is *strongly measurable*, if for any $x \in \mathscr{H}$, the map $t \mapsto T(t)x$ is Bochner measurable as an $\mathscr{H}$-valued function. The differences between these two definitions of measurability play an important role in defining solutions to Riccati equations and the way we approximate these equations. For example, let $S(t)$ be a C_0-semigroup over $\mathscr{H}$, then $t \mapsto S(t)x$ is norm continuous for every $x \in \mathscr{H}$, which implies that the map $S : I \to \mathscr{L}(\mathscr{H})$ is strongly measurable. However, $t \mapsto S(t)$ is Bochner measurable if and only if $t \mapsto S(t)$ is norm continuous for $t > 0$ (see Hille and Phillips book [39]). Therefore, if $t \mapsto S(t)$ is not norm continuous for $t > 0$, then the mapping $t \mapsto S(t)$ is not Bochner measurable and hence the Bochner integral $\int_0^1 S(t)\,\mathrm{d}t$ is **not** well-defined. However, one can define a bounded linear operator by

$$Vx = \int_0^1 S(t)x\,\mathrm{d}t$$

for each $x \in \mathscr{H}$, since $\|S(t)\|$ is uniformly bounded on $t \in [0, 1]$. This is often called the strong Bochner integral.

We recall the definitions of the standard Banach spaces $L^p(I; X)$. For $1 \le p < \infty$ the space $L^p(I; X)$ is defined to be the space of (equivalence classes) of *measurable functions* $f : I \to X$ such that $\|f\|_{L^p(I;X)} = (\int_I \|f(t)\|_X^p\,\mathrm{d}t)^{1/p} < \infty$, and $L^\infty(I; X)$ is define to be the space of (equivalence classes) of measurable functions such that $\|f\|_{L^\infty(I;X)} = \operatorname{ess\,sup}_{t\in I} \|f(t)\|_X < \infty$. When I is unbounded, then we can define the spaces $L^p_{loc}(I; X)$ as all (equivalence classes) of measurable functions such that their restriction to any compact interval $[a, b] \subset I$ belongs to $L^p(I; X)$, i.e., $f(\cdot) \in L^p_{loc}(I; X)$ if $(f\chi_{[a,b]})(\cdot) \in L^p(I; X)$ for any $[a, b] \subset I$ where $\chi_{[a,b]}$ is the characteristic function of the set $[a, b]$.

Let I be a compact interval. Since $\mathscr{I}_{p_1} \hookrightarrow \mathscr{I}_{p_2}$ for $1 \le p_1 \le p_2 \le \infty$, it follows that $\mathscr{C}(I; \mathscr{I}_{p_1}) \hookrightarrow \mathscr{C}(I; \mathscr{I}_{p_2})$. Then, $\mathscr{C}(I; \mathscr{I}_{p_1}) \hookrightarrow L^p(I; \mathscr{I}_{p_2})$ for all $1 \le p \le \infty$, and

$$\left(\int_I \|f(t)\|_{p_2}^p\,\mathrm{d}t\right)^{1/p} \le (m(I))^{1/p} \sup_{t\in I} \|f(t)\|_{p_1}.$$

Also, if $f(\cdot) \in L^p(I; \mathscr{I}_r)$ and $g(\cdot) \in L^q(I; \mathscr{I}_s)$, where $1/p + 1/q = 1$ and $1/r + 1/s = 1$, then $(fg)(\cdot)$ and $(gf)(\cdot)$ map I to $\mathscr{I}_1$ and they are Bochner measurable. This follows immediately by considering step functions f_n and g_n that converge point-wise a.e. to f and g in their respective norms. Since $f_n g_n$ and $g_n f_n$ are simple $\mathscr{I}_1$-valued and converge point-wise a.e. to fg and gf respectively. Finally, we note that

$$\int_I \|(fg)(t)\|_1 \,\mathrm{d}t \le \int_I \|f(t)\|_r \|g(t)\|_s \,\mathrm{d}t \le \left(\int_I \|f(t)\|_r^p \,\mathrm{d}t \right)^{1/p} \left(\int_I \|g(t)\|_s^q \,\mathrm{d}t \right)^{1/q},$$

and the same bound holds for $t \mapsto (gf)(t)$.

2. Smoothing Results. In this section we will study how multiplication of continuous mappings with values in $\mathscr{I}_p$ improve the continuity of strongly continuous $\mathscr{L}(\mathscr{H})$-valued mappings. Throughout this section we assume that $\mathbb{R}^+ = [0, \infty)$, $\mathscr{H}$ is a complex separable Hilbert space, $T : \mathbb{R}^+ \to \mathscr{L}(\mathscr{H})$ is strongly continuous (but not necessarily a C_0-semigroup of bounded linear operators) and that $\mathscr{I}^0$ is the set of nonzero finite rank operators.

Proposition 2.1. *Let $T : \mathbb{R}^+ \to \mathscr{L}(\mathscr{H})$ be a strongly continuous mapping and let $K \in \mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$, for some $1 \le p \le \infty$. Then, $t \mapsto T(t)K(t)$ and $t \mapsto K(t)T^*(t)$ belong to $\mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$.*

Proof. Suppose first that $K(t)$ is constant, i.e., $K(t) = K \in \mathscr{I}_p$ for $t \in \mathbb{R}^+$ and consider the case where $t \in [0, \tau]$ for a fixed finite $\tau > 0$. Since $K \in \mathscr{I}_p$ and $T(t) \in \mathscr{L}(\mathscr{H})$, it follows that $T(t)K \in \mathscr{I}_p$ for each $t \in [0, \tau]$.

We now prove that $t \mapsto T(t)K$ is $\mathscr{I}_p$-norm continuous when K is a rank one operator. If K be defined as $Kx = \langle \psi, x \rangle \varphi$ for some fixed $\psi, \varphi \in \mathscr{H}$, and all $x \in \mathscr{H}$, then $T(t)Kx = \langle \psi, x \rangle T(t)\varphi$. Let t and t_0 be in $[0, \tau]$ and $\{\phi_n\}_{n=1}^{\infty}$ an orthonormal basis of $\mathscr{H}$. Then, for $W \ne 0$ in $\mathscr{L}(\mathscr{H})$ of finite rank, the Cauchy-Schwartz inequality and Lemma 1.4 imply

$$\begin{aligned} |\mathrm{Tr}\,(W(T(t)K - T(t_0)K))\,| &\le \sum_{n=1}^{\infty} |\,\langle \phi_n, W(T(t) - T(t_0))K\phi_n \rangle\,| \\ &= \sum_{n=1}^{\infty} |\,\langle \psi, \phi_n \rangle\,|\;|\,\langle (\phi_n, W(T(t) - T(t_0))\varphi \rangle\,| \\ &\le \Big(\sum_{n=1}^{\infty} |\,\langle \psi, \phi_n \rangle\,|^2 \Big)^{1/2} \Big(\sum_{n=1}^{\infty} |\,\langle (\phi_n, W(T(t) - T(t_0))\varphi \rangle\,|^2 \Big)^{1/2} \\ &= \|\psi\| \|W\big(T(t)\varphi - T(t_0)\varphi\big)\| \le \|\psi\| \|W\|_q \|T(t)\varphi - T(t_0)\varphi\|, \end{aligned}$$

for any $1 \le q \le \infty$. Therefore, by Proposition 1.6, we have

$$\|T(t)K - T(t_0)K\|_p = \sup_{W \in \mathscr{I}^0} \frac{|\mathrm{Tr}\,\big(W\big(T(t)K - T(t_0)K\big)\big)\,|}{\|W\|_q} \le \|\psi\| \|T(t)\phi - T(t_0)\phi\|, \tag{2.1}$$

and hence $\|T(t)K - T(t_0)K\|_p \to 0$ as $t \to t_0$ because $t \mapsto T(t)$ is strongly continuous on $\mathbb{R}^+$. Therefore, $t \mapsto T(t)K$ is $\mathscr{I}_p$-norm continuous on $[0, \tau]$ when K is of rank one. We now use induction to establish the same result when K has finite rank.

Let $K = K_0 + K_1$ where the mapping $t \mapsto T(t)K_0$ continuous in the $\mathscr{I}_p$-norm and K_1 is of rank one. Since

$$\|T(t)K - T(t_0)K\|_p \le \|T(t)K_0 - T(t_0)K_0\|_p + \|T(t)K_1 - T(t_0)K_1\|_p,$$

it follows that $t \mapsto T(t)K$ is continuous in the $\mathscr{I}_p$-norm. Therefore, for any finite rank operator K, the map $t \mapsto T(t)K$ is $\mathscr{I}_p$-norm continuous on $[0, \tau]$.

Now we extend this result for any $K \in \mathscr{I}_p$. Let $K \in \mathscr{I}_p$ and $\{K_n\}_{n=1}^{\infty}$ be a sequence of finite rank operators such that $K_n \to K$ in the $\mathscr{I}_p$-norm (note that finite rank operators are dense in $\mathscr{I}_p$, in the corresponding norm). The triangle inequality implies

$$\|T(t)K - T(t_0)K\|_p \leq \|T(t_0)(K - K_n)\|_p + \|T(t)(K - K_n)\|_p + \|T(t)K_n - T(t_0)K_n\|_p,$$

and the Uniform Boundedness Theorem yields $\|T(t)\| \leq M_\tau$ for some $M_\tau > 0$ and for any $t \in [0, \tau]$. It now follows from Lemma 1.4 that

$$\|T(t)K - T(t_0)K\|_p \leq 2M_\tau \|K - K_n\|_p + \|T(t)K_n - T(t_0)K_n\|_p.$$

For any $\epsilon > 0$, there is an $N(\epsilon)$ such that $\|K - K_n\|_p < \epsilon/4M_\tau$ for $n \geq N(\epsilon)$. Since each K_n is of finite rank, for a fixed n, there is a $\delta > 0$ such that for $t \in (t_0 - \delta, t_0 + \delta) \cap [0, \tau]$ we observe $\|T(t)K_n - T(t_0)K_n\|_p < \epsilon/2$. Hence, it follows that for each $\epsilon > 0$

$$\|T(t)K - T(t_0)K\|_p < \epsilon,$$

for $t \in (t_0 - \delta, t_0 + \delta) \cap [0, \tau]$ for some $\delta = \delta(\epsilon)$. Therefore, if $K \in \mathscr{I}_p$, the map $t \mapsto T(t)K$ is $\mathscr{I}_p$-norm continuous on $[0, \tau]$. Moreover, since $\tau > 0$ was arbitrary, $t \mapsto T(t)K$ is $\mathscr{I}_p$-norm continuous on $\mathbb{R}^+$.

Finally, if $K \in \mathscr{I}_p$, then $K^* \in \mathscr{I}_p$ and $T(\cdot)K^* \in \mathscr{C}([0, \tau]; \mathscr{I}_p)$. Hence $KT^*(\cdot) \in \mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$ since $\|T(t)K^* - T(t_0)K^*\|_p = \|KT^*(t) - KT^*(t_0)\|_p$ and this completes the proof for the case when $K(t)$ is constant.

Suppose now that $K \in \mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$. If t and t_0 are in $[0, \tau]$ with $\tau > 0$ arbitrary, then $T(t)K(t)$ and $T(t_0)K(t_0)$ belong to $\mathscr{I}_p$. Again, the triangle inequality yields

$$\|T(t)K(t) - T(t_0)K(t_0)\|_p \leq \|T(t)\| \|K(t) - K(t_0)\|_p + \|T(t)K(t_0) - T(t_0)K(t_0)\|_p.$$

Since $K(\cdot) \in \mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$ and $t \mapsto \|T(t)\|$ is uniformly bounded in $[0, \tau]$ (by the Uniform Boundedness Principle), the first term in the right hand side goes to zero, as $t \to t_0$. The second term goes to zero since $K(t_0) \in \mathscr{I}_p$. Also, since $\|K^*(t)\|_p = \|K(t)\|_p$ the mapping $t \mapsto K^*(t)$ belongs to $\mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$ and hence $t \mapsto T(t)K^*(t)$ and $t \mapsto K^*(t)T^*(t)$ are both $\mathscr{I}_p$-norm continuous on $[0, \tau]$. Since $\tau > 0$ is arbitrary, the result holds on $\mathbb{R}^+$.
□

If the mapping $t \mapsto T(t)$ does not satisfy some additional property (for example, the semigroup property $T(t + s) = T(t)T(s)$ for $t, s > 0$ and $T(0) = I$) "strong continuity" can not be replaced by "weak continuity" in the previous proposition (for a counterexample check [52]). If the semigroup property and $T(0) = I$ are satisfied, then the strong continuity is implied by the weak continuity (for a proof, see [39] and [51]).

The previous propositions have stronger conclusions in the case where $t \mapsto T(t)$ is not just a strongly continuous mapping but a C_0-semigroup of linear operators over some Hilbert space $\mathscr{H}$. In this case, both $t \mapsto T(t)$ and $t \mapsto T^*(t)$ are strongly continuous. In fact, $T^*(t)$ is also a C_0-semigroup of linear operators over $\mathscr{H}$ as we can see in [51]. We summarize this in the following proposition.

Proposition 2.2. *Let $S(t)$ be a C_0-semigroup and let $K \in \mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$, for some $1 \leq p \leq \infty$. Then, the mappings: $t \mapsto S(t)K(t)$, $t \mapsto S^*(t)K(t)$, $t \mapsto K(t)S(t)$ and $t \mapsto K(t)S^*(t)$ belong to $\mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$.*

Proof. Since $S(t)$ is a C_0-semigroup over a Hilbert space, both mappings $t \mapsto S(t)$ and $t \mapsto S^*(t)$ are strongly continuous. The conclusion follows directly from Proposition 2.1. □

The next step is to show that multiplication by operators in $\mathscr{I}_p$ improves the convergence properties of approximating semigroups. In particular, we consider a sequence $\{S_n(t)\}$ of C_0-semigroups converging strongly (as $n \to \infty$) to a C_0-semigroup $S(t)$ uniformly on $[0, \tau]$. The following Lemma was proven for the case $p = 2$ by A. Germani, et al. in [35]. We will extend the results to any p satisfying $1 \le p \le \infty$ and to an arbitrary strongly continuous mapping $t \mapsto T(t)$.

LEMMA 2.3. *Let $\{T_n(t)\}$ be a sequence of strongly continuous $\mathscr{L}(\mathscr{H})$-valued functions and strongly convergent to $t \mapsto T(t)$ uniformly in $t \in [0, \tau]$ (i.e., the mapping $t \mapsto T_n(t)x$ is continuous for each $n \in \mathbb{N}$ and each $x \in \mathscr{H}$, and $\|T_n(t)x - T(t)x\| \to 0$ uniformly in $t \in [0, \tau]$ as $n \to \infty$). Assume that $1 \le p \le \infty$. If $\mathcal{K}$ is a compact set in $\mathscr{I}_p$, then*

$$\sup_{t \in [0,\tau]} \|T_n(t)K - T(t)K\|_p \to 0, \qquad \text{and} \qquad \sup_{t \in [0,\tau]} \|KT_n^*(t) - KT^*(t)\|_p \to 0,$$

both uniformly in $K \in \mathcal{K}$, as $n \to \infty$.

Proof. The Uniform Boundedness Principle implies that $\|T_n(t)\| \le M$ for some $M > 0$ and for all $n \in \mathbb{N}$ and $t \in [0, \tau]$. This in turn also implies that $\|T(t)\| \le M$. For each $t \in [0, \tau]$ we have that $\|T(t)x\| \le \sup_{n \in \mathbb{N}} \|T_n(t)x\| \le M\|x\|$ and hence $\|T(t)\| \le \sup_{n \in \mathbb{N}} \|T_n(t)\| \le M$.

If $K \in \mathscr{I}_p$, for each $t \in [0, \tau]$ and $n \in \mathbb{N}$, then $T_n(t)K$ and $T(t)K$ belong to $\mathscr{I}_p$ since the latter space is a double-sided ideal on $\mathscr{L}(\mathscr{H})$. We bound their difference by

$$\|T_n(t)K - T(t)K\|_p \le \|T_n(t)K\|_p + \|T(t)K\|_p \le 2M\|K\|_p.$$

Since $K \in \mathcal{K}$ and $\mathcal{K}$ is compact, it is bounded. Therefore, $\|T_n(t)K - T(t)K\|_p$ is uniformly bounded for all $t \in [0, \tau]$ and all $K \in \mathcal{K}$.

Define the functionals $J_n : \mathcal{K} \to \mathbb{R}$ by

$$J_n(K) := \sup_{t \in [0,\tau]} \|(T_n - T)(t)K\|_p,$$

which are uniformly bounded in $n \in \mathbb{N}$ and $K \in \mathcal{K}$.

Each mapping $t \mapsto T_n(t)$ is strongly continuous and hence $t \mapsto T(t)$ is strongly continuous since it is the strong limit of $\{T_n(\cdot)\}$ uniformly in $t \in [0, \tau]$. This follows from the inequality

$$\|T(t)x - T(s)x\| \le \|T(t)x - T_n(t)x\| + \|T(s)x - T_n(s)x\| + \|T_n(t)x - T_n(s)x\|,$$

and the fact that each $t \mapsto T_n(t)x$ is continuous in $[0, \tau]$ and $\|T(t)x - T_n(t)x\| \to 0$ as $n \to \infty$, uniformly in $t \in [0, \tau]$.

Proposition 2.1 implies that $t \mapsto (T_n - T)(t)K$ is $\mathscr{I}_p$-norm continuous if $K \in \mathscr{I}_p$. For the sake of brevity, we define $|||F(\cdot)|||_p = \sup_{t \in [0,\tau]} \|F(t)\|_p$, for $F \in \mathscr{C}([0, \tau]; \mathscr{I}_p)$. Thus, if K_1 and K_2 are arbitrary elements of $\mathcal{K}$, we have that

$$\begin{aligned} |J_n(K_1) - J_n(K_2)| &= \big| |||(T_n - T)(\cdot)K_1|||_p - |||(T_n - T)(\cdot)K_2|||_p \big| \\ &\le |||(T_n - T)(\cdot)(K_1 - K_2)|||_p \le 2M\|K_1 - K_2\|_p. \end{aligned}$$

Hence, for each $n \in \mathbb{N}$, $K \mapsto J_n(K)$ is a uniformly continuous mapping on the compact set $\mathcal{K}$ and therefore attains its maximum over $\mathcal{K}$, i.e., $\sup_{K \in \mathcal{K}} J_n(K) = J_n(\hat{K}^n)$, for some $\hat{K}^n \in \mathcal{K}$.

Since J_n is uniformly bounded in $\mathcal{K}$, define $\epsilon := \overline{\lim}_{n\to\infty}(\sup_{K \in \mathcal{K}} J_n(K)) = \overline{\lim}_{n\to\infty} J_n(\hat{K}^n)$, where $\{\hat{K}^n\}_{n=1}^{\infty}$ is the sequence of maximizers defined above. Then, there is a subsequence $J_{n_j}(\hat{K}^{n_j})$ for such that $J_{n_j}(\hat{K}^{n_j}) \to \epsilon$ as $j \to \infty$. Without loss of generality, suppose that $\epsilon = \lim_{n\to\infty} J_n(\hat{K}^n)$. Also, since $\mathcal{K}$ is compact, the sequence $\{\hat{K}^n\}_{n=1}^{\infty}$ contains a convergent subsequence, and for the sake of brevity suppose $\hat{K}^n \to \hat{K}$ as $n \to \infty$, for some $\hat{K} \in \mathcal{K}$. Since we have already established the inequality $|J_n(\hat{K}^n) - J_n(\hat{K})| \le 2M\|\hat{K}^n - \hat{K}\|_p$, then it follows that $\epsilon = \lim_{n\to\infty} J_n(\hat{K})$.

Now we prove that $\epsilon = 0$. Assume first that $\hat{K}$ is of rank one, defined by $\hat{K}x = \langle \psi, x\rangle\, \phi$ (for some ψ and ϕ, and all x in $\mathscr{H}$). If $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathscr{H}$ and $\mathscr{I}^0$ is the set of nonzero finite rank operators, we have

$$\begin{aligned}
J_n(\hat{K}) &= \sup_{t\in[0,\tau]} \|(T_n - T)(t)\hat{K}\|_p \\
&= \sup_{t\in[0,\tau]} \left(\sup_{W\in\mathscr{I}^0} \frac{|\mathrm{Tr}(W(T_n - T)(t)\hat{K})|}{\|W\|_q} \right) \\
&\le \sup_{t\in[0,\tau]} \left(\sup_{W\in\mathscr{I}^0} \frac{\sum_{n=1}^{\infty} |\langle \phi_n, W(T_n - T)(t)\hat{K}\phi_n\rangle|}{\|W\|_q} \right) \\
&= \sup_{t\in[0,\tau]} \left(\sup_{W\in\mathscr{I}^0} \frac{\sum_{n=1}^{\infty} |\, \langle \psi, \phi_n\rangle \,| \quad |\, \langle \phi_n, W(T_n - T)(t)\varphi\rangle \,|}{\|W\|_q} \right) \\
&\le \sup_{t\in[0,\tau]} \left(\sup_{W\in\mathscr{I}^0} \frac{\left(\sum_{n=1}^{\infty} |\, \langle \psi, \phi_n\rangle \,|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |\, \langle \phi_n, W(T_n - T)(t)\varphi\rangle \,|^2\right)^{1/2}}{\|W\|_q} \right) \\
&= \sup_{t\in[0,\tau]} \left(\sup_{W\in\mathscr{I}^0} \frac{\|\psi\|\|W(T_n - T)(t)\varphi\|}{\|W\|_q} \right) \\
&\le \|\psi\| \sup_{t\in[0,\tau]} \left(\sup_{W\in\mathscr{I}^0} \frac{\|W\|_q\|(T_n - T)(t)\varphi\|}{\|W\|_q} \right) \\
&\le \|\psi\| \sup_{t\in[0,\tau]} \|(T_n - T)(t)\phi\|.
\end{aligned}$$

This implies $J_n(\hat{K}) \to 0$ as $n \to \infty$ because $T_n(t)\phi \to T(t)\phi$ uniformly in $t \in [0,\tau]$ as $n \to \infty$. Next, suppose that $\hat{K} = K_1 + K_2$, such that $\lim_{n\to\infty} J_n(K_1) = 0$ and K_2 is of rank one. Since $J_n(K_1 + K_2) \le J_n(K_1) + J_n(K_2)$, we observe that $\lim_{n\to\infty} J_n(\hat{K}) = 0$ and hence this is valid for all $\hat{K}$ of finite rank. Finally, suppose that $\hat{K} \in \mathscr{I}_p$. Then, there is a sequence $\{K_m\}_{m=1}^{\infty}$ of finite rank operators such that $\|\hat{K} - K_m\|_p \to 0$ as $m \to \infty$, and

$$J_n(\hat{K}) \le J_n(\hat{K} - K_m) + J_n(K_m) \le 2M\|\hat{K} - K_m\|_p + J_n(K_m).$$

Hence $\overline{\lim}_{n\to\infty} J_n(\hat{K}) \le 2M\|\hat{K} - K_m\|_p$, for any $m \in \mathbb{N}$ and therefore $\epsilon = \lim_{n\to\infty} J_n(\hat{K}) = 0$ for any $\hat{K} \in \mathscr{I}_p$. Thus we have proven that

$$\epsilon = \lim_{n\to\infty} \sup_{K\in\mathcal{K}} J_n(K) = \lim_{n\to\infty} \sup_{K\in\mathcal{K}} \|T_n(t)K - T(t)K\|_p = 0.$$

In order to prove the second part of the initial statement, define $\mathcal{K}^* = \{K^* : K \in \mathcal{K}\}$. Then $\mathcal{K}^*$ is also a compact subset of $\mathscr{I}_p$. Therefore, $\|T_n(t)K^* - T(t)K^*\|_p \to 0$ uniformly in $K \in \mathcal{K}$ and $t \in [0,\tau]$, but $\|T_n(t)K^* - T(t)K^*\|_p = \|(T_n(t)K^* - T(t)K^*)^*\|_p = \|KT_n^*(t) - KT^*(t)\|_p$ and so $\sup_{[0,\tau]} \|KT_n^*(t) - KT^*(t)\|_p \to 0$ uniformly in $K \in \mathcal{K}$. This completes the proof. □

It is well known that if $\{T_n(t)\}$ is a sequence of strongly continuous $\mathscr{L}(\mathscr{H})$-valued functions and strongly convergent to $t \mapsto T(t)$ uniformly in $t \in [0, \tau]$, this does not imply (in general) that the sequence of mappings $\{T_n^*(t)\}$ is strongly convergent to $t \mapsto T^*(t)$ uniformly in $t \in [0, \tau]$. This assertion even fails in the case of C_0-semigroups (see [12] for a counterexample). However, in the case where one has convergence of the dual maps, we have the following result.

Lemma 2.4. *Let $\{T_n(t)\}$ and $\{T_n^*(t)\}$ be sequences of $\mathscr{L}(\mathscr{H})$-valued functions, strongly continuous and strongly convergent to the maps $t \mapsto T(t)$ and $t \mapsto T^*(t)$, respectively and uniformly in $t \in [0, \tau]$. Suppose also that $1 \leq p \leq \infty$ and let $\mathcal{K}$ be a compact set in $\mathscr{I}_p$. Then as $n \to \infty$,*

$$\sup_{K \in \mathcal{K}} \left(\sup_{t \in [0,\tau]} \|G_n(t, T, K)\|_p \right) \to 0,$$

where $G_n(t, T, K)$ is any of the following: $t \mapsto (T_n(t)K - T(t)K)$, $t \mapsto (KT_n^*(t) - KT^*(t))$, $t \mapsto (KT_n(t) - KT(t))$ or $t \mapsto (T_n^*(t)K - T^*K(t))$.

Proof. The proof follows by the application of the previous lemma. □

3. The Integral Riccati Equation. In this section we focus on the Riccati integral equation

$$\Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)(BB^* - \Sigma(s)(C^*C)(s)\Sigma(s))S^*(t-s)\,\mathrm{d}s, \qquad (3.1)$$

where its integral term will be shown to be well-defined as a Bochner integral in Theorem 3.1 . Unlike much of the existing literature in which this equation is considered in the mild sense, we shall interpret (3.1) by employing the Bochner integral with operator-valued integrand. This is suitable for our applications. The advantage of considering the integrand in equation (3.1) as Bochner integrable is that it can be approximated by step functions. Therefore, its integral (for a fixed t) can be uniformly approximated by finite sums of operators and this could be applied to the development of numerical methods.

In general, the integral term in (3.1) is *not* Bochner integrable due to the fact that the map $t \mapsto S(t)$ is not necessarily norm continuous (see section 1.3). However, here is where compactness plays a key role: the results of section 2 imply that the strong measurability $t \mapsto S(t)$ is enough to determine that $t \mapsto S(t)K$, $t \mapsto S^*(t)K$, $t \mapsto KS(t)$ and $t \mapsto KS^*(t)$ belong to $\mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$ if $K \in \mathscr{I}_p$. This is the key result to improve the measurability of the integrand in (3.1).

Throughout this section we assume that $\mathscr{H}$ be a separable complex Hilbert space, $I = [0, \tau]$ or $I = \mathbb{R}^+ = [0, \infty)$ and $1 \leq p \leq \infty$. For the sake of brevity, we define the mappings F and G by

$$F(\cdot) := BB^*(\cdot) \qquad \text{and} \qquad G(\cdot) := C^*C(\cdot),$$

and study properties of (3.1) from properties of F and G. The motivation of considering these maps as time dependent can be seen in the motivating example in section 1.1.

3.1. Properties of the Mapping γ. We will define the right hand side of (3.1) as $\gamma(\Sigma(\cdot))$ and hence fixed points of γ are solutions to the integral Riccati equation. We will now prove that γ is a well defined function in the appropriate spaces.

Theorem 3.1. *Let $S(t)$ be a C_0-semigroup over $\mathscr{H}$, and suppose that*

(i) $\Sigma_0 \in \mathscr{I}_p$
(ii) $F(\cdot) \in L^1_{loc}(I; \mathscr{I}_p)$
(iii) $G(\cdot) \in L^\infty_{loc}(I; \mathscr{L}(\mathscr{H}))$
If $\Sigma(\cdot) \in L^2_{loc}(I; \mathscr{I}_{2p})$*, then for all* $t \in I$ *the mapping*

$$s \mapsto S(t-s)\big(F - \Sigma G \Sigma\big)(s) S^*(t-s) \tag{3.2}$$

is Bochner integrable as a $\mathscr{I}_p$*-valued mapping on* $[0,t]$ *and* $\gamma(\Sigma)(\cdot)$ *defined by*

$$\gamma(\Sigma(\cdot))(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)\big(F - \Sigma G \Sigma\big)(s) S^*(t-s)\, \mathrm{d}s, \tag{3.3}$$

is a well defined function $\gamma : L^2_{loc}(I; \mathscr{I}_{2p}) \to \mathscr{C}(I; \mathscr{I}_p)$*. Moreover, since* $\gamma(L^2_{loc}(I; \mathscr{I}_{2p})) \subset \mathscr{C}(I; \mathscr{I}_p)$*, it follows that* $\mathscr{C}(I; \mathscr{I}_p)$ *is an* γ*-invariant subspace of* $L^2_{loc}(I; \mathscr{I}_{2p})$*.*

If instead of **(iii)***,* $G(\cdot)$ *satisfies the stronger condition*
(iii') $G(\cdot) \in L^\infty_{loc}(I; \mathscr{I}_p)$,
and $\Sigma(\cdot) \in L^2_{loc}(I; \mathscr{L}(\mathscr{H}))$*, then* (3.2) *is again Bochner integrable as a* $\mathscr{I}_p$*-valued mapping on* $[0,t]$ *and* $\gamma(\Sigma)(\cdot) \in \mathscr{C}(I; \mathscr{I}_p)$*. In this case,* $\gamma(L^2_{loc}(I; \mathscr{L}(\mathscr{H}))) \subset \mathscr{C}(I; \mathscr{I}_p)$*, and* $\mathscr{C}(I; \mathscr{I}_p)$ *is a* γ*-invariant subspace of* $L^2_{loc}(I; \mathscr{L}(\mathscr{H}))$*.*

Observe that since $\mathscr{I}_p \subset \mathscr{I}_{2p} \subset \mathscr{I}_\infty$ for any $1 \leq p \leq \infty$, we observe that $\mathscr{C}(I; \mathscr{I}_p) \subset L^2_{loc}(I; \mathscr{I}_{2p})$. For $p = 1$ this implies that $\mathscr{C}(I; \mathscr{I}_1)$ is continuously embedded in $L^2(I; \mathscr{I}_2)$ (with I compact) and the latter is a Hilbert space. Therefore, if we can find a locally square integrable, Hilbert-Schmidt valued solution of the Riccati equation, that function is trace class-valued and continuous in trace norm. The other very important feature to observe is that if **(iii)** holds, then it is not possible to define γ over $L^2_{loc}(I; \mathscr{L}(\mathscr{H}))$. The reason for this is that $S(t)$ is a general C_0-semigroup (and not necessarily norm continuous for $t > 0$). That is, $t \mapsto S(t)$ is Bochner measurable as a $\mathscr{L}(\mathscr{H})$-valued mapping if and only if it is operator-norm continuous for $t > 0$ (see [39]).

Proof. [Proof of Theorem 3.1] Since $S(t)$ is a C_0-semigroup of linear operators on the Hilbert space $\mathscr{H}$, then $S^*(t)$ is also a C_0-semigroup on the same Hilbert space $\mathscr{H}$ and the map $t \mapsto S^*(t)$ is strongly continuous. Since $\Sigma_0 \in \mathscr{I}_p$, we have $S(\cdot)\Sigma_0 \in \mathscr{C}(I; \mathscr{I}_p)$, and since $t \mapsto S^*(t)$ is strongly continuous it follows that $S(\cdot)\Sigma_0 S^*(\cdot) \in \mathscr{C}(I; \mathscr{I}_p)$ by Proposition 2.2.

Suppose that $t \in I$ is fixed. We begin by proving that the mapping $s \mapsto S(t-s)F(s)S^*(t-s)$ is Bochner measurable on $[0,t]$. Suppose first that $F(\cdot)$ is a characteristic function, i.e., $F(s) = f\,\chi_E(s)$ with $f \in \mathscr{I}_p$ and $E \subseteq [0,t]$ measurable. Hence, $S(t-s)F(s)S^*(t-s) = S(t-s) f S^*(t-s)\,\chi_E(s)$ is Bochner measurable (for $s \in [0,t]$) since it is the product of a $\mathscr{I}_p$-valued continuous function and a scalar measurable function (see [1]). By linearity, $s \mapsto S(t-s)F(s)S^*(t-s)$ is measurable for any $F : [0,t] \mapsto \mathscr{I}_p$ which is simple. If $F(\cdot) \in L^1(I; \mathscr{I}_p)$, it is measurable and there is a sequence of simple functions $F_n(\cdot)$ such that $\|F(s) - F_n(s)\|_p \to 0$ a.e. for $s \in [0,t]$ as $n \to \infty$. Since $S(t)$ is a C_0-semigroup, there is an M_t such that $\|S(t-s)\| \leq M_t$ for all $s \in [0,t]$ and we have

$$\|S(t-s)F(s)S^*(t-s) - S(t-s)F_n(s)S^*(t-s)\|_p \leq M_t^2 \|F(s) - F_n(s)\|_p.$$

Therefore, we conclude that $s \mapsto S(t-s)F(s)S^*(t-s)$ is Bochner measurable on $[0,t]$ as a $\mathscr{I}_p$-valued function, for any Bochner measurable function $F : I \to \mathscr{I}_p$ since is the sequence of measurable functions $s \mapsto S(t-s)F_n(s)S^*(t-s)$.

Suppose again that $t \in I$ is fixed and that **(iii)** holds. Based on the above paragraph, to prove that the mapping $s \mapsto S(t-s)\Sigma(s)G(s)\Sigma(s)S^*(t-s)$ is Bochner measurable for $s \in [0,t]$ as a $\mathscr{I}_p$-valued function, we only need to prove that $s \mapsto \Sigma(s)G(s)\Sigma(s)$ is Bochner measurable for $\Sigma(\cdot) \in L^2_{loc}(I;\mathscr{I}_{2p})$ (or $\Sigma(\cdot) \in L^2_{loc}(I;\mathscr{L}(\mathscr{H}))$ when **(iii')** holds). First consider $\sigma_1, \sigma_2 \in \mathscr{I}_{2p}$, $c \in \mathscr{L}(\mathscr{H})$ and E_1, E_2, E_3 measurable subsets of $[0,t]$, then

$$\big(\sigma_1 \chi_{E_1}(s)\big)\big(c\chi_{E_3}(s)\big)\big(\sigma_2\chi_{E_4}(s)\big) = (\sigma_1 c \sigma_2)\, \chi_{\bigcap_{i=1}^3 E_i}(s)$$

is Bochner measurable as a $\mathscr{I}_p$-valued function. We conclude this because $\sigma_1 c\sigma_2 \in \mathscr{I}_p$ ($\sigma_1 \in \mathscr{I}_{2p}$, we have $\sigma_1 c \in \mathscr{I}_{2p}$ and hence $(\sigma_1 c)\sigma_2 \in \mathscr{I}_p$, since $\sigma_2 \in \mathscr{I}_{2p}$ by Lemma 1.4) and $E_1 \cap E_2 \cap E_3$ is measurable (the same holds if $\sigma_i \in \mathscr{L}(\mathscr{H})$ and $c \in \mathscr{I}_p$). By the distributive law, $s \mapsto \Sigma(s)G(s)\Sigma(s)$ is $\mathscr{I}_p$-valued, Bochner measurable when $s \mapsto \Sigma(s)$ is a simple $\mathscr{I}_{2p}-$valued, and $s \mapsto G(s)$ is a simple $\mathscr{L}(\mathscr{H})$-valued (or when $s \mapsto \Sigma(s)$ is simple $\mathscr{L}(\mathscr{H})$-valued, and $s \mapsto G(s)$ is simple $\mathscr{I}_p-$valued). If $\Sigma(\cdot) \in L^2_{loc}(I;\mathscr{I}_{2p})$ and $G(\cdot) \in L^\infty_{loc}(I;\mathscr{L}(\mathscr{H}))$ there are sequences of simple functions $\Sigma_n(\cdot)$ and $G_n(\cdot)$, $\mathscr{I}_{2p}$-valued and $\mathscr{L}(\mathscr{H})$-valued respectively, that converge point-wise a.e. in $s \in [0,t]$ (in the corresponding norm) to $\Sigma(\cdot)$ and $G(\cdot)$. For each $s \in [0,t]$, we have $\Sigma(s)G(s)\Sigma(s) \in \mathscr{I}_p$, hence (suppressing "$(s)$" for the sake of brevity) from the equality

$$\Sigma G\Sigma - \Sigma_n G_n \Sigma_n = (\Sigma - \Sigma_n)G\Sigma + \Sigma_n\Big(G(\Sigma-\Sigma_n) + (G-G_n)\Sigma_n\Big)$$

it follows that

$$\begin{aligned}&\|(\Sigma G\Sigma - \Sigma_n G_n\Sigma_n)(s)\|_p \leq \\ &\|(\Sigma-\Sigma_n)(s)\|_{2p}\Big(\|G(s)\|\|\Sigma(s)\|_{2p} + \|\Sigma_n(s)\|_{2p}\|G(s)\|\Big) + \|(G-G_n)(s)\|\|\Sigma_n(s)\|^2_{2p}.\end{aligned}$$

If **(iii')** holds instead, for $\Sigma(\cdot) \in L^2_{loc}(I;\mathscr{L}(\mathscr{H}))$ and $G(\cdot) \in L^\infty_{loc}(I;\mathscr{I}_p)$ there are sequences of simple functions $\Sigma_n(\cdot)$ and $G_n(\cdot)$, $\mathscr{L}(\mathscr{H})$-valued and $\mathscr{I}_p$-valued respectively, that converge point-wise a.e. in $s \in [0,t]$ (in the corresponding norm) to $\Sigma(\cdot)$ and $G(\cdot)$. In this case, we obtain the inequality

$$\begin{aligned}&\|(\Sigma G\Sigma - \Sigma_n G_n\Sigma_n)(s)\|_p \leq \\ &\|(\Sigma-\Sigma_n)(s)\|\Big(\|G(s)\|_p\|\Sigma(s)\| + \|\Sigma_n(s)\|\|G(s)\|_p\Big) + \|(G-G_n)(s)\|_p\|\Sigma_n(s)\|^2.\end{aligned}$$

Hence, we have that $s \mapsto \Sigma(s)G(s)\Sigma(s)$ is a $\mathscr{I}_p$-valued, Bochner measurable function since it is the point-wise limit a.e. of measurable functions when **(iii)** or **(iii')** hold. Therefore, $s \mapsto S(t-s)\Sigma(s)G(s)\Sigma(s)S^*(t-s)$ is $\mathscr{I}_p$-valued (with fixed $t \in I$) and Bochner measurable for $s \in [0,t]$ by the results of the previous paragraph.

We have proven that the integrand in the definition of the operator γ is Bochner measurable. Now we prove that the integrand is locally Bochner integrable (note that for Banach space X, $f \in L^1(I;X)$ iff f is Bochner measurable and $\int_I \|f(t)\|_X \,\mathrm{d}t < \infty$, see [1]). Recall that if $A_1 \in \mathscr{I}_p$, $A_2 \in \mathscr{I}_{2p}$ and $A \in \mathscr{L}(\mathscr{H})$, this implies that $A_iA, AA_i \in \mathscr{I}_{ip}$ for $i = 1,2$ and $\|A_iA\|_{ip}$ and $\|AA_i\|_{ip}$ are bounded above by $\|A\|\|A_i\|_{ip}$ (see Lemma 1.4). By using these properties of $\mathscr{I}_p$ and $\mathscr{I}_{2p}$, we obtain the inequality

$$\int_0^t \|S(t-s)F(s)^*S(t-s)\|_p \,\mathrm{d}s \leq M_t^2 \int_0^t \|F(s)\|_p \,\mathrm{d}s = M_t^2 \|F(\cdot)\|_{L^1([0,t];\mathscr{I}_p)}.$$

If case **(iii)** holds and $\Sigma(\cdot) \in L^2_{loc}(I;\mathscr{I}_{2p})$, then we have

$$\begin{aligned}\int_0^t \|S(t-s)\big(\Sigma G\Sigma\big)(s)S^*(t-s)\|_p \,\mathrm{d}s &\le M_t^2 \int_0^t \|\big(\Sigma G\Sigma\big)(s)\|_p \,\mathrm{d}s \\ &\le M_t^2 \int_0^t \|G(s)\|\|\Sigma(s)\|^2_{2p} \,\mathrm{d}s \\ &\le M_t^2 \|G(\cdot)\|_{L^\infty([0,t];\mathscr{L}(\mathscr{H}))}\|\Sigma(\cdot)\|^2_{L^2([0,t];\mathscr{I}_{2p})},\end{aligned}$$

and if **(iii')** holds and $\Sigma(\cdot) \in L^2_{loc}(I;\mathscr{L}(\mathscr{H}))$, then

$$\begin{aligned}\int_0^t \|S(t-s)\big(\Sigma G\Sigma\big)(s)S^*(t-s)\|_p \,\mathrm{d}s &\le M_t^2 \int_0^t \|\big(\Sigma G\Sigma\big)(s)\|_p \,\mathrm{d}s \\ &\le M_t^2 \int_0^t \|G(s)\|_p\|\Sigma(s)\|^2 \,\mathrm{d}s \\ &\le M_t^2 \|G(\cdot)\|_{L^\infty([0,t];\mathscr{I}_p)}\|\Sigma(\cdot)\|^2_{L^2([0,t];\mathscr{L}(\mathscr{H}))}.\end{aligned}$$

This implies the local Bochner integrability of the integrands in any case.

We have established the local integrability of the integrand in the definition (3.3) of γ. Now we prove that the integral defines a continuous $\mathscr{I}_p$-valued function. Assume that $I = [0,\tau]$ with $\tau > 0$ arbitrary. Define $H : I \times I \to \mathscr{I}_p$ by

$$H(t,s) = \chi_{[0,t)}(s)\Big(S(t-s)\big(F - \Sigma G\Sigma\big)(s)S^*(t-s)\Big),$$

so that

$$H(t,s) = \begin{cases} S(t-s)\big(F-\Sigma G\Sigma\big)(s)S^*(t-s), & 0 \le s < t \le \tau; \\ 0, & 0 \le t \le s \le \tau\ . \end{cases}$$

We have that $\gamma(\Sigma)(t) = S(t)\Sigma_0 S^*(t) + \int_0^\tau H(t,s)\,\mathrm{d}s$ when $t \in [0,\tau]$.

It follows from above that for each fixed $t \in I = [0,\tau]$, the mapping $s \mapsto H(t,s)$ is Bochner measurable in $[0,\tau]$. In addition, we have the bound

$$\|H(t,s)\|_p \le M_\tau^2 \|\big(F - \Sigma G\Sigma\big)(s)\|_p.$$

Thus, the right hand side is integrable and independent of t. In order to complete the proof we show that

$$\lim_{t_n \to t} \|H(t_n,s) - H(t,s)\|_p = 0,$$

a.e. in $s \in [0,\tau]$. To prove this, select a fixed $s \in (0,\tau)$ and observe $\big(F - \Sigma G\Sigma\big)(s) \in \mathscr{I}_p$. Hence $t \mapsto S(t-s)\big(F - \Sigma G\Sigma\big)(s)S^*(t-s)$ is $\mathscr{I}_p$-continuous for $t \in (s,\tau)$ by Proposition 2.1 and then $t \mapsto H(t,s)$ is $\mathscr{I}_p$-continuous for $t \in (s,\tau)$. If $t \in (0,s)$, then $H(t,s) = 0$. Hence, $H(t_n,s) \to H(t,s)$ as $t_n \to t$ a.e. in $s \in [0,\tau]$. Finally let $t_n \to t$, by Dominated Convergence Theorem we observe that $\int_0^\tau H(t_n,s)\,\mathrm{d}s \to \int_0^\tau H(t,s)\,\mathrm{d}s$, i.e., the mapping

$$t \mapsto \int_0^t S(t-s)\big(F - \Sigma G\Sigma\big)(s)S^*(t-s)\,\mathrm{d}s,$$

is continuous in $\mathscr{I}_p$-norm on $t \in [0,\tau]$ with $\tau > 0$ arbitrary.

We have proven that if $\Sigma(\cdot) \in L^2([0,\tau];\mathscr{I}_{2p})$ when **(iii)** holds (or if $\Sigma(\cdot) \in L^2([0,\tau];\mathscr{L}(\mathscr{H}))$ when **(iii')** holds), we conclude that $\gamma(\Sigma)(\cdot) \in \mathscr{C}([0,\tau];\mathscr{I}_p)$. Since $\tau > 0$ is arbitrary, the conclusion of the theorem follows. □

3.2. Uniformly Continuous Semigroups and Approximations. We will prove in this section that there are solutions to $\Sigma = \gamma(\Sigma)$ in the case when $S(t)$ is a uniformly continuous semigroup. We require the following definition of monotonically increasing operator valued mappings.

Definition 3.2. *Let $t \mapsto T(t)$ be a point-wise non-negative (and hence self-adjoint) mapping defined as $T : I \to \mathscr{L}(\mathscr{H})$, where $I \subset \mathbb{R}$ is some interval and $\mathscr{H}$ is a Hilbert space. We say that $t \mapsto T(t)$ is monotonically increasing if $T(t_1) \le T(t_2)$ (that is $T(t_2) - T(t_1) \ge 0$) whenever $t_1 \le t_2$ and $t_1, t_2 \in I$.*

We rely on the following result whose proof is provided in Appendix B.

Lemma 3.3. *Suppose that $E(\cdot) \in L^\infty_{loc}(\mathbb{R}^+; \mathscr{L}(\mathscr{H}))$, $D(\cdot) \in \mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$, and also that both are point-wise non-negative. In addition, suppose that $t \mapsto D(t)$ is monotonically increasing. Then there is a unique solution of*

$$\Sigma(t) = D(t) - \int_0^t \Sigma(s)E(s)\Sigma^*(s)\,\mathrm{d}s, \tag{3.4}$$

in $\mathscr{C}(\mathbb{R}^+; \mathscr{L}(\mathscr{H}))$ which also belongs to $\mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$. Even more, the solution $t \mapsto \Sigma(t)$ satisfies that $\Sigma^*(t) = \Sigma(t) \ge 0$ for all $t \in \mathbb{R}^+$ and

$$\sup_{t\in[0,\tau]} \|\Sigma(t)\|_p \le \|D(\tau)\|_p, \tag{3.5}$$

for any $\tau > 0$.

Although, we are still several steps away of being able to consider the general case, Lemma 3.3 can be apply to some Riccati equations. For example, consider

$$\Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)\big(BB^* - \Sigma(C^*C)\Sigma\big)(s)S^*(t-s)\,\mathrm{d}s,$$

with $S(t) = I$. Then, the previous equation is

$$\Sigma(t) = \left(\Sigma_0 + \int_0^t (BB^*)(s)\,\mathrm{d}s\right) - \int_0^t (\Sigma(C^*C)\Sigma^*)(s)\,\mathrm{d}s. \tag{3.6}$$

If $\Sigma_0^* = \Sigma_0 \ge 0$ (and $\Sigma_0 \in \mathscr{I}_p$), $BB^*(\cdot) \in L^1_{loc}(\mathbb{R}^+; \mathscr{I}_p)$ and $C^*C(\cdot) \in L^\infty(\mathbb{R}^+; \mathscr{L}(\mathscr{H}))$ (with $BB^*(\cdot)$ and $C^*C(\cdot)$ point-wise non-negative), then $t \mapsto \Sigma_0 + \int_0^t (BB^*)(s)\,\mathrm{d}s$ is a monotonically increasing non-negative mapping and hence we observe a unique solution $t \mapsto \Sigma(t)$ of the Riccati equation (3.6) in $\mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$. In addition, $\Sigma(\cdot)$ is point-wise self-adjoint, non-negative, and bounded on compact intervals $[0, \tau]$ by

$$\sup_{t\in[0,\tau]} \|\Sigma(t)\| \le \|\Sigma_0\|_p + \|BB^*(\cdot)\|_{L^1([0,\tau];\mathscr{I}_p)}.$$

Now, we need to turn our attention to the case then $S(t)$ is a different semigroup of linear operators than the identity. We can prove the following.

Theorem 3.4. *Let $S(t)$ be a uniformly continuous semigroup on $\mathscr{H}$ such that $\|S(t)\| \le Me^{\omega t}$ for $t \in \mathbb{R}^+$. Additionally, suppose that*

(i) *$\Sigma_0 \in \mathscr{I}_p$ and $\Sigma_0 \ge 0$.*

(ii) *$F(\cdot) \in L^1_{loc}(\mathbb{R}^+; \mathscr{I}_p)$, where $t \mapsto F(t)$ is point-wise non-negative.*

(iii) *$G(\cdot) \in L^\infty_{loc}(\mathbb{R}^+; \mathscr{L}(\mathscr{H}))$, with $t \mapsto G(t)$ is point-wise non-negative.*

Then, the equation

$$\Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)\big(F - \Sigma G \Sigma\big)(s) S^*(t-s)\,\mathrm{d}s,$$

has a unique solution in $L^2_{loc}(\mathbb{R}^+; \mathscr{I}_{2p})$*, which verifies also to belong to* $\mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$ *and* $\Sigma^*(t) = \Sigma(t) \geq 0$ *for* $t \in \mathbb{R}^+$*. More over, we have*

$$\|\Sigma(t)\|_p \leq M^2 e^{2\omega t}\left(\|\Sigma_0\|_p + M^2 e^{2\omega t}\int_0^t \|F(s)\|_p\,\mathrm{d}s\right),$$

for $t \in \mathbb{R}^+$*.*

Proof. Since $S(t)$ is uniformly continuous, $S(t) = e^{At}$ for some $A \in \mathscr{L}(\mathscr{H})$, which implies that we can embed $S(t)$ and $S^*(t)$ in groups of linear operators. So $S(t) = e^{At}$ and $S^*(t) = e^{A^*t}$ for $t \in \mathbb{R}$. Then the maps $t \mapsto S(t)$ and $t \mapsto S^*(t)$ have $\mathbb{R}$ as domain, are continuous in operator norm and satisfy the group property: $S(t)S(s) = S(t+s)$ and $S^*(t)S^*(s) = S^*(t+s)$ for all $-\infty < s, t < \infty$.

Since $F(\cdot) \in L^1_{loc}(\mathbb{R}^+; \mathscr{I}_p)$, then $\hat{F}(t) := S(-t)F(t)S^*(-t)$ for $t \in \mathbb{R}^+$, satisfies that $\hat{F}(\cdot) \in L^1([0,\tau]; \mathscr{I}_p)$ for any $\tau > 0$. The measurability follows immediately since $t \mapsto S(-t)$ and $t \mapsto S^*(-t)$ are norm continuous, and the local integrability follows from the bound $\|\hat{F}(t)\|_p \leq M^2 e^{2\omega t}\|F(t)\|_p$. Similarly, since $G(\cdot) \in L^\infty_{loc}(\mathbb{R}^+; \mathscr{L}(\mathscr{H}))$, then $\hat{G}(t) := S(t)G(t)S^*(t)$ for $t \in \mathbb{R}^+$ satisfies that $\hat{G}(\cdot) \in L^\infty([0,\tau]; \mathscr{L}(\mathscr{H}))$ for arbitrary $\tau > 0$. We also observe that $\hat{F}^*(t) = \hat{F}(t) \geq 0$ and $\hat{G}^*(t) = \hat{G}(t) \geq 0$. Hence, the equation

$$\Pi(t) = \left(\Sigma_0 + \int_0^t \hat{F}(s)\,\mathrm{d}s\right) - \int_0^t \Pi(s)\hat{G}(s)\Pi^*(s)\,\mathrm{d}s, \tag{3.7}$$

by Lemma 3.3, has a unique solution in $\mathscr{C}([0,\tau]; \mathscr{L}(\mathscr{H}))$, that also belongs to $\mathscr{C}([0,\tau]; \mathscr{I}_p)$, and such that $\Pi^*(t) = \Pi(t) \geq 0$. This follows since $t \mapsto \Sigma_0 + \int_0^t \hat{F}(s)\,\mathrm{d}s$ is a monotonic, point-wise self-adjoint and non-negative mapping. We also observe $\|\Pi(t)\|_p \leq \|\Sigma_0\|_p + \int_0^t \|\hat{F}(s)\|_p\,\mathrm{d}s$.

Define $\Sigma(t) = S(t)\Pi(t)S^*(t)$. By definition $\Sigma(\cdot) \in \mathscr{C}([0,\tau]; \mathscr{I}_p)$ by Proposition 2.2 and it is also point-wise non-negative and self-adjoint since $\Pi^*(t) = \Pi(t) \geq 0$. Then, this implies that

$$
\begin{aligned}
\Sigma(t) &= S(t)\Pi(t)S^*(t) \\
&= S(t)\left(\Sigma_0 + \int_0^t \hat{F}(s)\,\mathrm{d}s\right)S^*(t) - S(t)\int_0^t \Pi(s)\hat{G}(s)\Pi(s)\,\mathrm{d}s\ \ S^*(t) \\
&= S(t)\Sigma_0 S^*(t) + \int_0^t S(t)\hat{F}(s)S^*(t)\,\mathrm{d}s - \int_0^t S(t)\Pi(s)\hat{G}(s)\Pi(s)S^*(t)\,\mathrm{d}s \\
&= S(t)\Sigma_0 S^*(t) + \int_0^t S(t)S(-s)F(s)S^*(-s)S^*(t)\,\mathrm{d}s \\
&\quad - \int_0^t S(t)S(-s)\Sigma(s)S^*(-s)S^*(s)G(s)S(s)S(-s)\Sigma(s)S^*(-s)S^*(t)\,\mathrm{d}s \\
&= S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)F(s)S^*(t-s)\,\mathrm{d}s \\
&\qquad\qquad - \int_0^t S(t-s)\Sigma(s)G(s)\Sigma(s)S^*(t-s)\,\mathrm{d}s,
\end{aligned}
$$

this is $t \mapsto \Sigma(t)$ satisfies the desired Riccati equation on $t \in [0,\tau]$ with $\tau > 0$ arbitrary.

Suppose there is another solution $t \mapsto \tilde{\Sigma}(t) \in \mathscr{C}([0,\tau];\mathscr{I}_p)$ to this equation. Since the adjoint map $A \mapsto A^*$ is a bounded (conjugate) linear map on $\mathscr{L}(\mathscr{H})$, it follows that $\left(\int_0^t Y(s)\,\mathrm{d}s\right)^* = \int_0^t Y^*(s)\,\mathrm{d}s$ for any Bochner integrable $\mathscr{L}(\mathscr{H})$-valued function $Y(\cdot)$. This implies, since $\Sigma_0, G(\cdot)$ and $F(\cdot)$ are point-wise non-negative (and hence self-adjoint) that $t \mapsto \tilde{\Sigma}^*(t) \in \mathscr{C}([0,\tau];\mathscr{I}_p)$ solves the same Riccati equation. A direct application of Gronwall's lemma over the difference $\|(\tilde{\Sigma} - \tilde{\Sigma}^*)(\cdot)\|$ implies that $\tilde{\Sigma}(\cdot)$ is point-wise self-adjoint. Then define $t \mapsto \tilde{\Pi}(t) = S(-t)\tilde{\Sigma}(t)S^*(-t) \in \mathscr{C}([0,\tau];\mathscr{I}_p)$ and this a solution to the equation (3.7). Since $t \mapsto \Pi(t)$ was the unique solution to (3.7), hence $\tilde{\Pi}(t) = \Pi(t)$, and then $\tilde{\Sigma}(t) = \Sigma(t)$ since $t \mapsto S(t)$ and $t \mapsto S^*(t)$ are invertible for each $t \in \mathbb{R}$.

Suppose there is another solution $t \mapsto \bar{\Sigma}(t)$ of the Riccati equation belonging to $L^2([0,\tau];\mathscr{I}_{2p})$. Then, by Theorem 3.1, $\bar{\Sigma}(\cdot) \in \mathscr{C}([0,\tau];\mathscr{I}_p)$ and then $\bar{\Sigma}(t) = \Sigma(t)$ for all $t \in [0,\tau]$ by the previous paragraph.

The inequality $\|\Pi(t)\|_p \leq \|\Sigma_0\|_p + \int_0^t \|\hat{F}(s)\|_p\,\mathrm{d}s$ immediately leads to

$$
\|\Sigma(t)\|_p \leq M^2 e^{2\omega t}\left(\|\Sigma_0\|_p + M^2 e^{2\omega t}\int_0^t \|F(s)\|_p\,\mathrm{d}s\right),
$$

for $t \in \mathbb{R}^+$, since $\|\Sigma(t)\|_p \leq M^2 e^{2\omega t}\|\Pi(t)\|_p$ and $\|\hat{F}(t)\|_p \leq M^2 e^{2\omega t}\|F(t)\|_p$. □

It seems we are one step away from proving existence and uniqueness of the $\mathscr{I}_p$-norm continuous solution to the integral Riccati equation when $S(t)$ is a C_0-semigroup. But several difficulties arise if we try to apply the same idea in the previous proofs for this case. Fortunately, we can overcome this problem, by using the aid of local and approximation results as we will prove subsequently. In Theorem 3.4, we have proven that

$$
\Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)\big(F - \Sigma G \Sigma\big)(s)S^*(t-s)\,\mathrm{d}s, \tag{3.8}
$$

has a unique solution in $\mathscr{C}(\mathbb{R}^+;\mathscr{I}_p)$ when $F(\cdot) \in L^1_{loc}(\mathbb{R}^+;\mathscr{I}_p)$, $G(\cdot) \in L^\infty_{loc}(\mathbb{R}^+;\mathscr{L}(\mathscr{H}))$ and $S(t)$ is a uniformly continuous semigroup. Also, by Theorem 3.1, if $\Sigma(\cdot) \in$

$\mathscr{C}(\mathbb{R}^+;\mathscr{I}_p)$, then $\gamma(\Sigma)$ is well defined when $S(t)$ is a C_0-semigroup. Now, we will prove that if we have a solution to the equation (3.8) when $S(t)$ is a C_0-semigroup, then under certain hypotheses we will be able to approximate this solution by solutions to equation (3.8) when $S_n(t)$ is a sequence of uniformly continuous semigroups.

THEOREM 3.5. *Suppose that $S(t)$ is a C_0-semigroup of linear operators over $\mathscr{H}$, and that $\{S_n(t)\}$ is a sequence of uniformly continuous semigroups over the same Hilbert space $\mathscr{H}$ that satisfy, for each $x \in \mathscr{H}$,*

$$\|S(t)x - S_n(t)x\| \to 0 \qquad \text{and} \qquad \|S^*(t)x - S_n^*(t)x\| \to 0,$$

as $n \to \infty$, uniformly in compact intervals. Suppose also the following.

(i) *$\Sigma_0 \geq 0$ and the sequence $\{\Sigma_0^n\}_{n=1}^\infty$ are in $\mathscr{I}_p$, $\Sigma_0^n \geq 0$ for all $n \in \mathbb{N}$ and $\|\Sigma_0 - \Sigma_0^n\|_p \to 0$ as $n \to \infty$.*

(ii) *$BB^*(\cdot)$ and the sequence $\{F_n(\cdot)\}_{n=1}^\infty$ are in $L^1_{loc}(\mathbb{R}^+;\mathscr{I}_p)$, $BB^*(t) \geq 0$ and $F_n(t) \geq 0$ for all $t \in \mathbb{R}^+$ and all $n \in \mathbb{N}$ and satisfy*

$$\int_0^\tau \|BB^*(t) - F_n(s)\|_p \,\mathrm{d}s \to 0,$$

for any fixed $\tau > 0$ and as $n \to \infty$.

(iii) *$C^*C(\cdot)$ and the sequence $\{G_n(\cdot)\}_{n=1}^\infty$ are in $L^\infty(\mathbb{R}^+;\mathscr{L}(\mathscr{H}))$, $C^*C(t) \geq 0$ and $G_n(t) \geq 0$ for all $t \in \mathbb{R}^+$ and all $n \in \mathbb{N}$ and satisfy*

$$\operatorname*{ess\,sup}_{t\in[0,\tau]} \|C^*C(t) - G_n(t)\| \to 0,$$

for any fixed $\tau > 0$ and as $n \to \infty$.

Then, if $\Sigma(\cdot) \in \mathscr{C}([0,a],\mathscr{I}_p)$ is a solution of

$$\Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)\big(BB^* - \Sigma(C^*C)\Sigma\big)(s) S^*(t-s)\,\mathrm{d}s,$$

for some $a > 0$ and if $\Sigma_n(\cdot) \in \mathscr{C}(\mathbb{R}^+,\mathscr{I}_p)$ is the sequence of solutions of

$$\Sigma_n(t) = S_n(t)\Sigma_0^n S_n^*(t) + \int_0^t S_n(t-s)\big(F_n - \Sigma_n G_n \Sigma_n\big)(s) S_n^*(t-s)\,\mathrm{d}s,$$

we observe that

$$\sup_{t\in[0,a]} \|\Sigma(t) - \Sigma_n(t)\|_p \to 0,$$

as $n \to \infty$.

Proof. First note that since $S_n(t)$ is a uniformly continuous semigroup for each $n \in \mathbb{N}$, the sequence of solutions of the Riccati equation $t \mapsto \Sigma_n(t)$ belong to $\mathscr{C}(\mathbb{R}^+;\mathscr{I}_p)$ according to Theorem 3.4.

Suppose $\tau > 0$ is fixed. The convergence of the sequences $\{\Sigma_0^n\}_{n=1}^\infty$, $\{F_n(\cdot)\}_{n=1}^\infty$ and $\{G_n(\cdot)\}_{n=1}^\infty$ (in the respective norms) imply that they are bounded, and hence there are positive numbers σ_0, b_τ and c_τ such that

$$\begin{aligned}
\sigma_0 &= \sup_{n\in\mathbb{N}} \|\Sigma_0^n\|_p \geq \|\Sigma_0\|_p,\\
b_\tau &= \sup_{n\in\mathbb{N}} \|F_n(\cdot)\|_{L^1([0,\tau];\mathscr{I}_p)} \geq \|BB^*(\cdot)\|_{L^1([0,\tau];\mathscr{I}_p)}\\
c_\tau &= \sup_{n\in\mathbb{N}} \|G_n(\cdot)\|_{L^\infty([0,\tau];\mathscr{L}(\mathscr{H}))} \geq \|C^*C(\cdot)\|_{L^\infty([0,\tau];\mathscr{L}(\mathscr{H}))}.
\end{aligned}$$

Also, the Uniform Boundedness Principle implies that there is a constant M_τ such that

$$\sup(\|S(t)\|, \|S_n(t)\|) \leq M_\tau,$$

where the sup ranges in all $n \in \mathbb{N}$ and all $t \in [0, \tau]$ (and of course the same bound is valid for $S^*(t)$ and $S_n^*(t)$).

The sequence $\{\Sigma_n(\cdot)\}_{n=1}^\infty$ is bounded, as we observed in Theorem 3.4, as

$$\begin{aligned}\|\Sigma_n(t)\|_p &\leq M_\tau^2\Big(\|\Sigma_0^n\|_p + M_\tau^2 \int_0^\tau \|F_n(s)\|_p \,\mathrm{d}s\Big)\\ &\leq M_\tau^2(\sigma_0 + M_\tau^2 \tau b_\tau),\end{aligned}$$

in $t \in [0, \tau]$. We define $\rho_\tau = M_\tau^2(\sigma_0 + M_\tau^2 \tau b_\tau)$.

We first prove that $S_n(t)\Sigma_0^n S_n^*(t) \to S(t)\Sigma_0 S^*(t)$ in the sup norm for $\mathscr{C}([0, \tau]; \mathscr{I}_p)$. We have the following bound

$$\begin{aligned}&\|S_n(t)\Sigma_0^n S_n^*(t) - S(t)\Sigma_0 S^*(t)\|_p \leq \\ &\|S_n(t)(\Sigma_0^n - \Sigma_0)S_n^*(t)\|_p + \|S_n(t)\Sigma_0(S_n^*(t) - S^*(t))\|_p + \|(S_n(t) - S(t))\Sigma_0 S^*(t)\|_p.\end{aligned} \tag{3.9}$$

The first term in the right hand side satisfies

$$\|S_n(t)(\Sigma_0^n - \Sigma_0)S_n^*(t)\|_p \leq M_\tau^2 \|\Sigma_0^n - \Sigma_0\|_p,$$

and then converges to zero since $\|\Sigma_0^n - \Sigma_0\|_p \to 0$. The second term in the right hand side of inequality in (3.9) satisfies

$$\|S_n(t)\Sigma_0(S_n^*(t) - S^*(t))\|_p \leq M_\tau \sup_{t\in[0,\tau]} \|\Sigma_0(S_n^*(t) - S^*(t))\|_p,$$

and by Lemma 2.3, goes to zero as $n \to \infty$ and uniformly in $t \in [0, \tau]$ because $\Sigma_0 \in \mathscr{I}_p$ and due to the strong convergence of $S_n^*(t)$ to $S^*(t)$. Since $t \mapsto S(t)$ is strongly continuous and $\Sigma_0 \in \mathscr{I}_p$, then by Proposition 2.1 we observe $t \mapsto \Sigma_0 S^*(t)$ is $\mathscr{I}_p$-norm continuous which implies that the set $\{\Sigma_0 S^*(t) / \, t \in [0, \tau]\}$ is compact in the $\mathscr{I}_p$-norm topology. Then again by Lemma 2.3, we observe $\sup_{t\in[0,\tau]} \|(S_n(t) - S(t))\Sigma_0 S^*(t)\|_p \to 0$ as $n \to \infty$. Therefore

$$\sup_{t\in[0,\tau]} \|S_n(t)\Sigma_0^n S_n^*(t) - S(t)\Sigma_0 S^*(t)\|_p \to 0,$$

as $n \to \infty$.

Next, we prove that the mapping $t \mapsto \int_0^t S_n(t-s)F_n(s)S_n^*(t-s)\,\mathrm{d}s$ converges to $t \mapsto \int_0^t S(t-s)BB^*(s)S^*(t-s)\,\mathrm{d}s$ in the sup $\mathscr{I}_p$-norm. Both mappings are elements of $\mathscr{C}([0, \tau]; \mathscr{I}_p)$ as we have proven in the first part of the proof of Theorem 3.6. Then, we observe the following bound on the integrands

$$\begin{aligned}&\|S(t-s)BB^*(s)S^*(t-s) - S_n(t-s)F_n(s)S_n^*(t-s)\|_p \leq \\ &\quad \|S_n(t-s)(F_n - BB^*)(s)S_n^*(t-s)\|_p + \|S_n(t-s)BB^*(s)(S_n^* - S^*)(t-s)\|_p + \\ &\quad \|(S_n - S)(t-s)BB^*(s)S^*(t-s)\|_p.\end{aligned} \tag{3.10}$$

For the first term in the right hand side, we observe that

$$\begin{aligned}\sup_{t\in[0,\tau]}\int_0^t \|S_n(t-s)(F_n-BB^*)(s)S_n^*(t-s)\|_p\,\mathrm{d}s &\le\\ &\le \sup_{t\in[0,\tau]}\int_0^t \|S_n(t-s)\|\|(F_n-BB^*)(s)\|_p\|S_n^*(t-s)\|\,\mathrm{d}s\\ &\le M_\tau^2\sup_{t\in[0,\tau]}\int_0^t \|(F_n-BB^*)(s)\|_p\,\mathrm{d}s\\ &\le M_\tau^2\|(F_n-BB^*)(\cdot)\|_{L^1([0,\tau];\mathscr{I}_p)}.\end{aligned}$$

Hence, it goes to zero by the initial hypotheses. For the second term in the right hand side of the inequality in (3.10), we proceed as follows. Since $BB^*(\cdot)\in L^1([0,\tau];\mathscr{I}_p)$, it can be approximated with simple $\mathscr{I}_p$-valued functions. Suppose that $F(\cdot)$ is simple. Then, we have the following bound

$$\begin{aligned}&\|S_n(t-s)BB^*(s)(S_n^*-S^*)(t-s)\|_p\le\\ &\quad\le \|S_n(t-s)(BB^*-F)(s)(S_n^*-S^*)(t-s)\|_p+\|S_n(t-s)F(s)(S_n^*-S^*)(t-s)\|_p\\ &\quad\le 2M_\tau^2\|(BB^*-F)(s)\|_p+M_\tau\|F(s)(S_n^*-S^*)(t-s)\|_p.\end{aligned}$$

We know that F is of the form $F(t)=\sum^N f_k\chi_{E_k}(t)$, with a finite number of nonzero $f_k\in\mathscr{I}_p$. The set $\mathcal{K}_F=\{f_k/\ \ 1\le k\le N\}$ is compact in the topology of $\mathscr{I}_p$. Therefore

$$\begin{aligned}\sup_{s\in[0,t]}\|F(s)(S_n^*-S^*)(t-s)\|_p &\le \sup_{f_k\in\mathcal{K}_F}\sup_{s\in[0,t]}\|f_k(S_n^*-S^*)(t-s)\|_p\\ &\le \sup_{f_k\in\mathcal{K}_F}\sup_{t\in[0,\tau]}\|f_k(S_n^*-S^*)(t)\|_p,\end{aligned}$$

and the right hand side goes to zero by Lemma 2.3. Then, in order to clarify things, let $\epsilon>0$ be arbitrary, and choose a simple $\mathscr{I}_p$-valued function F_ϵ such that $\|(BB^*-F_\epsilon)(\cdot)\|_{L^1([0,\tau];\mathscr{I}_p)}<\frac{\epsilon}{4M_\tau^2}$. Also there is an $N(\epsilon)>0$ such that if $n\ge N(\epsilon)$, then $\|K(S_n^*-S^*)(t)\|_p<\frac{\tau\epsilon}{2M_\tau}$ uniformly in $t\in[0,\tau]$ and $K\in\mathcal{K}_{F_\epsilon}$. Therefore

$$\begin{aligned}&\sup_{t\in[0,\tau]}\int_0^t \|S_n(t-s)BB^*(s)(S_n^*-S^*)(t-s)\|_p\,\mathrm{d}s\le\\ &\quad\le \sup_{t\in[0,\tau]}\left(2M_\tau^2\int_0^t\|(BB^*-F_\epsilon)(s)\|_p\,\mathrm{d}s+M_\tau\int_0^t\|F(s)(S_n^*-S^*)(t-s)\|_p\,\mathrm{d}s\right)\\ &\quad\le 2M_\tau^2\|(BB^*-F_\epsilon)(\cdot)\|_{L^1([0,\tau];\mathscr{I}_p)}+\tau M_\tau\sup_{K\in\mathcal{K}_{F_\epsilon},\,t\in[0,\tau]}\|K(S_n^*-S^*)(t)\|_p\\ &\quad<\epsilon.\end{aligned}$$

The same argument shows that for any $\epsilon>0$, there is an $N(\epsilon)>0$ such that if $n\ge N(\epsilon)$, we observe

$$\sup_{t\in[0,\tau]}\int_0^t\|(S_n-S)(t-s)BB^*(s)S^*(t-s)\|_p\,\mathrm{d}s<\epsilon.$$

Since $\epsilon>0$ is arbitrary, this proves that the mapping $t\mapsto\int_0^t S_n(t-s)F_n(s)S_n^*(t-s)\,\mathrm{d}s$ converges to $t\mapsto\int_0^t S(t-s)BB^*(s)S^*(t-s)\,\mathrm{d}s$ in the $\mathscr{C}([0,\tau];\mathscr{I}_p)$ norm.

Let $I = [0,\tau] = [0,a]$. We observe (when $0 \le s \le t \in I$) the following bound

$$\begin{aligned}\|S(t-s)\big(\Sigma(C^*C)\Sigma\big)(s)S^*(t-s) - S_n(t-s)\big(\Sigma_n G_n \Sigma_n\big)(s)S_n^*(t-s)\|_p \le \\ \|S_n(t-s)(\Sigma_n G_n \Sigma_n - \Sigma(C^*C)\Sigma)(s)S_n^*(t-s)\|_p + \\ \|S_n(t-s)\big(\Sigma(C^*C)\Sigma\big)(s)(S_n^* - S^*)(t-s)\|_p + \\ \|(S_n - S)(t-s)\big(\Sigma(C^*C)\Sigma\big)(s)S^*(t-s)\|_p.\end{aligned}$$

Since $\Sigma(\cdot) \in \mathscr{C}(I;\mathscr{I}_p)$ and $C^*C(\cdot) \in L^\infty([0,\tau];\mathscr{L}(\mathscr{H}))$, then it is straightforward to observe that $\Sigma(C^*C)\Sigma(\cdot) \in L^\infty(I;\mathscr{I}_p)$ and therefore $(\Sigma(C^*C)\Sigma)(\cdot) \in L^1(I;\mathscr{I}_p)$. Hence it can be approximated by simple $\mathscr{I}_p$-valued functions. Therefore, as we proved before, for each $\epsilon > 0$, there is an $N(\epsilon)$ such that if $n \ge N(\epsilon)$, then

$$\sup_{t\in[0,\tau]} \int_0^t \|S_n(t-s)(\Sigma(C^*C)\Sigma)(s)(S_n^* - S^*)(t-s)\|_p \,\mathrm{d}s < \epsilon,$$

and

$$\sup_{t\in[0,\tau]} \int_0^t \|(S_n - S)(t-s)(\Sigma(C^*C)\Sigma)(s)S^*(t-s)\|_p \,\mathrm{d}s < \epsilon.$$

As we did before, suppressing "(t)" for the sake of brevity, we observe the following bound:

$$\begin{aligned}\|\Sigma(C^*C)\Sigma - \Sigma_n G_n \Sigma_n\|_p \le \\ \|\Sigma - \Sigma_n\|_p \Big(\|C^*C\|\|\Sigma\|_p + \|\Sigma_n\|_p\|C^*C\|\Big) + \|C^*C - G_n\|\|\Sigma_n\|_p^2.\end{aligned}$$

We know that $\sup_{t\in[0,\tau]} \|C^*C(t)\| \le c_\tau$, $\sup_{t\in[0,\tau]} \|\Sigma_n(t)\|_p \le \rho_\tau$ and we define $\hat{\rho} = \sup_{t\in[0,\tau]} \|\Sigma(t)\|_p$ and $\rho = \max(\rho_\tau, \hat{\rho})$. Hence

$$\begin{aligned}\int_0^t \|S_n(t-s)(\Sigma_n G_n \Sigma_n - \Sigma(C^*C)\Sigma)(s)S_n^*(t-s)\|_p \,\mathrm{d}s \le \\ M_\tau^2 c_\tau \rho \int_0^t \|(\Sigma - \Sigma_n)(s)\|_p \,\mathrm{d}s + \tau\rho^2 \|(C^*C - G_n)(\cdot)\|_{L^\infty([0,\tau];\mathscr{L}(\mathscr{H}))}.\end{aligned}$$

Finally, define the following functions

$$\begin{aligned}h_1(n) &= \sup_{t\in[0,\tau]} \|S_n(t)\Sigma_0^n S_n^*(t) - S(t)\Sigma_0 S^*(t)\|_p, \\ h_2(n) &= \sup_{t\in[0,\tau]} \int_0^t \|S_n(t-s)(BB^* - F_n)(s)S_n^*(t-s)\|_p \,\mathrm{d}s, \\ h_3(n) &= \sup_{t\in I} \Bigg(\int_0^t \|S_n(t-s)(\Sigma(C^*C)\Sigma)(s)(S_n^* - S^*)(t-s)\|_p \,\mathrm{d}s + \\ &\qquad \int_0^t \|(S_n - S)(t-s)(\Sigma(C^*C)\Sigma)(s)S^*(t-s)\|_p \,\mathrm{d}s \Bigg), \\ h_4(n) &= \tau\rho^2 \|(C^*C - G_n)(\cdot)\|_{L^\infty([0,\tau];\mathscr{L}(\mathscr{H}))}.\end{aligned}$$

We have thus far shown that $\lim_{n\to\infty} h_i(n) = 0$, independently of $t \in I$, for $i = 1,2,3$ and $\lim_{n\to\infty} h_4(n) = 0$, by initial hypotheses. Therefore, since $\Sigma(\cdot)$ and $\Sigma_n(\cdot)$ satisfy the Riccati equation, the difference is bounded as

$$\|(\Sigma - \Sigma_n)(t)\|_p \leq h(n) + 2M_\tau^2 c_\tau \rho \int_0^t \|(\Sigma - \Sigma_n)(s)\|_p \, \mathrm{d}s,$$

in $t \in I$ and with $h(n) = \sum_{k=1}^4 h_k(n)$. Then a direct application of Grönwall's Lemma implies that

$$\sup_{t \in I} \|(\Sigma - \Sigma_n)(t)\|_p \leq h(n) e^{2M_\tau^2 c_\tau \rho \, m(I)},$$

where $m(I)$ is the measure of $I = [0, a] = [0, \tau]$ and because $n \mapsto h(n)$ is independent of t. Finally, since $\lim_{n\to\infty} h(n) = 0$, the Theorem is proved. ☐

3.3. Solutions for C_0-Semigroups. Let $S(t)$ be a C_0-semigroup over $\mathscr{H}$, $F(\cdot) \in L^1_{loc}(\mathbb{R}^+; \mathscr{I}_p)$ and $G(\cdot) \in L^\infty_{loc}(\mathbb{R}^+; \mathscr{L}(\mathscr{H}))$. Suppose there is a solution $\Sigma(\cdot) \in \mathscr{C}([0, a]; \mathscr{I}_p)$, of

$$\Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)\big(F - \Sigma G \Sigma\big)(s) S^*(t-s) \, \mathrm{d}s, \tag{3.11}$$

and that $\{S_n(t)\}_{n=1}^\infty$ is a sequence of uniformly continuous semigroups such that $S_n(t)$ and $S_n^*(t)$ converge strongly, and uniformly in $t \in [0, a]$, to $S(t)$ and $S^*(t)$ as $n \to \infty$, respectively. Then, Theorem 3.5, implies that the sequence of solutions $\{\Sigma_n(\cdot)\}_{n=1}^\infty$ in $\mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$ of

$$\Sigma_n(t) = S_n(t)\Sigma_0 S_n^*(t) + \int_0^t S_n(t-s)\big(F - \Sigma_n G \Sigma_n\big)(s) S_n^*(t-s) \, \mathrm{d}s, \tag{3.12}$$

satisfies $\sup_{t \in [0,a]} \|\Sigma(t) - \Sigma_n(t)\|_p \to 0$ as $n \to \infty$. In the next result we prove existence of a local solution $\Sigma(\cdot) \in \mathscr{C}([0, a]; \mathscr{I}_p)$ to (3.11) and use the sequence of solutions 3.12 to extend the local solution to the entire interval $\mathbb{R}^+$.

An interesting feature of the following result is that, under certain conditions, there is a unique solution to (3.11) in $L^2([0, \tau]; \mathscr{I}_2)$ that also belongs to $\mathscr{C}([0, \tau]; \mathscr{I}_1)$. This is an useful result for approximation purposes because $L^2([0, \tau]; \mathscr{I}_2)$ is a Hilbert space and $\mathscr{C}([0, \tau]; \mathscr{I}_1)$ is not.

THEOREM 3.6. *Let $\mathscr{H}$ be a separable complex Hilbert space, $I = [0, \tau]$ or $I = \mathbb{R}^+$, $S(t)$ be a C_0-semigroup on $\mathscr{H}$, and suppose that*
(i) $\Sigma_0 \in \mathscr{I}_p$ *and* $\Sigma_0 \geq 0$*;*
(ii) $BB^*(\cdot) \in L^1_{loc}(I; \mathscr{I}_p)$*, with* $BB^*(t) \geq 0$ *for* $t \in I$*;*
(iii) $C^*C(\cdot) \in L^\infty_{loc}(I; \mathscr{L}(\mathscr{H}))$*, with* $C^*C(t) \geq 0$ *for* $t \in I$*.*
Then, the equation

$$\Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)\big(BB^* - \Sigma(C^*C)\Sigma\big)(s) S^*(t-s) \, \mathrm{d}s,$$

where the integral is a Bochner integral, has a unique solution in the space $L^2_{loc}(I; \mathscr{I}_{2p})$, and even more the solution belongs to $\mathscr{C}(I; \mathscr{I}_p)$ and is point-wise self-adjoint and non-negative.

Proof. The following argument will be a modification of the one of Da Prato in [9] and [23]. The idea of the proof consists in an application of the Contraction Mapping Principle to prove existence and uniqueness locally, and then making use of the non-negativity of the solution to extend existence and uniqueness to the entire interval I.

Using the definition of the map γ in Theorem 3.1, we can write the Riccati equation as $\Sigma = \gamma(\Sigma)$. Since $\gamma : L^2_{loc}(I; \mathscr{I}_{2p}) \to \mathscr{C}(I; \mathscr{I}_p)$, it is enough to search for fixed points of γ in the latter space, because $\mathscr{C}(I; \mathscr{I}_p) \subset L^2_{loc}(I; \mathscr{I}_{2p})$.

Let $\tau > 0$ be such that $[0, \tau] \subset I$. Define $b_\tau = \|BB^*(\cdot)\|_{L^1([0,\tau];\mathscr{I}_p)}$ and $c_\tau = \|(C^*C)(\cdot)\|_{L^\infty([0,\tau];\mathscr{L}(\mathscr{H}))}$ and suppose that $\sup_{t\in[0,\tau]} \|\Sigma(t)\|_p \le \rho$. Then,

$$\begin{aligned}
\|\gamma(\Sigma)(t)\|_p &\le \|S(t)\Sigma_0 S^*(t)\|_p + \int_0^t \|S(t-s)\big(BB^* - \Sigma(C^*C)\Sigma\big)(s)S^*(t-s)\|_p \,\mathrm{d}s \\
&\le M_\tau^2 \Big(\|\Sigma_0\|_1 + b_\tau + t c_\tau \sup_{t\in[0,\tau]} \|\Sigma(t)\|_p^2 \Big) \\
&\le M_\tau^2 \Big(\|\Sigma_0\|_p + b_\tau + t c_\tau \, \rho^2 \Big),
\end{aligned}$$

where $\|S(t)\| = \|S^*(t)\| \le M_\tau$ for all $t \in [0, \tau]$. Note that $M_\tau \ge 1$ exist since $S(t)$ and $S^*(t)$ are C_0-semigroups. Hence, taking the supremum over $[0, \tau_0]$ with $0 < \tau_0 \le \tau$,

$$\sup_{t\in[0,\tau_0]} \|\gamma(\Sigma)(t)\|_p \le M_\tau^2 \Big(\|\Sigma_0\|_p + b_\tau + \tau_0 c_\tau \, \rho^2 \Big). \tag{3.13}$$

Now, let $\Lambda_i(\cdot) \in \mathscr{C}([0, \tau]; \mathscr{I}_p)$ and $\sup_{t\in[0,\tau]} \|\Lambda_i(t)\|_p \le \rho$ for $i = 1, 2$, then we obtain the following bounds for the difference $(\gamma(\Lambda_1) - \gamma(\Lambda_1))(t)$ when $t \in [0, \tau_0]$ as

$$\begin{aligned}
\|\gamma(\Lambda_1)(t) - \gamma(\Lambda_2)(t)\|_p &\le \int_0^t \|S(t-s)\big(\Lambda_2(C^*C)\Lambda_2 - \Lambda_1(C^*C)\Lambda_1\big)(s)S^*(t-s)\|_p \,\mathrm{d}s \\
&\le M_\tau^2 \int_0^t \|\big((\Lambda_2 - \Lambda_1)(C^*C)\Lambda_2 + \Lambda_1(C^*C)(\Lambda_2 - \Lambda_1)\big)(s)\|_p \,\mathrm{d}s \\
&\le M_\tau^2 \int_0^t \|\big((\Lambda_2 - \Lambda_1)(C^*C)\Lambda_2\big)(s)\|_p + \|\big(\Lambda_1(C^*C)(\Lambda_2 - \Lambda_1)\big)(s)\|_p \,\mathrm{d}s \\
&\le M_\tau^2 c_\tau \int_0^t \|(\Lambda_2 - \Lambda_1)(s)\|_p \|\Lambda_2(s)\|_p + \|\Lambda_1(s)\|_p \|(\Lambda_2 - \Lambda_1)(s)\|_p \,\mathrm{d}s \\
&\le M_\tau^2 c_\tau \tau_0 \sup_{s\in[0,\tau_0]} \Big(\|\Lambda_1(s)\|_p + \|\Lambda_2(s)\|_p \Big) \sup_{t\in[0,\tau_0]} \|(\Lambda_1 - \Lambda_2)(t)\|_p.
\end{aligned}$$

Hence

$$\sup_{t\in[0,\tau_0]} \|\big(\gamma(\Lambda_1) - \gamma(\Lambda_2)\big)(t)\|_p \le 2M_\tau^2 c_\tau \tau_0 \rho \sup_{t\in[0,\tau_0]} \|(\Lambda_1 - \Lambda_2)(t)\|_p. \tag{3.14}$$

Then, define β and ρ and choose $0 < \tau_0 \le \tau$ such that

$$\begin{aligned}
&\beta = M_\tau^2(\|\Sigma_0\|_p + b_\tau); &\qquad& b_\tau + \tau_0 \rho^2 c_\tau \le \beta; \\
&\rho = 2M_\tau^2 \beta &\qquad& 2M_\tau^2 c_\tau \tau_0 \rho \le \frac{1}{2};
\end{aligned}$$

which is always possible since $b_\tau < \beta$ (if $\|\Sigma_0\|_p = 0$ use $M_\tau > 1$). Therefore, since $\|\Sigma_0\|_p \le \beta$ by definition, we observe from that (3.13) and (3.14), that

$$\sup_{t\in[0,\tau_0]} \|\gamma(\Sigma)(t)\|_p \le \rho \quad \text{and} \quad \sup_{t\in[0,\tau_0]} \|\big(\gamma(\Lambda_1) - \gamma(\Lambda_2)\big)(t)\|_p \le \frac{1}{2} \sup_{t\in[0,\tau_0]} \|(\Lambda_1 - \Lambda_2)(t)\|_p.$$

Therefore, the mapping γ defines a contraction on the ball

$$\mathbf{B}_{s,\rho} = \Big\{ F(\cdot) \in \mathscr{C}([0,\tau_0];\mathscr{I}_p) : \sup_{t\in[0,\tau_0]} \|F(t)\|_p \leq \rho \Big\},$$

and then the equation $\Sigma = \gamma(\Sigma)$ defines an unique solution on $\mathbf{B}_{s,\rho}$ by the Contraction Mapping Theorem.

Since $S(t)$ is a C_0-semigroup over $\mathscr{H}$, there is a sequence $\{S_n(t)\}_{n=1}^{\infty}$ of uniformly continuous semigroups over the same Hilbert space $\mathscr{H}$ that satisfy that for each $x \in \mathscr{H}$,

$$\|S(t)x - S_n(t)x\| \to 0 \qquad \text{and} \qquad \|S^*(t)x - S_n^*(t)x\| \to 0,$$

as $n \to \infty$, uniformly in $t \in [0,\tau]$: Since $S(t)$ is a C_0-semigroup over a Hilbert space $\mathscr{H}$, then $S^*(t)$ also is a C_0-semigroup over $\mathscr{H}$ and even more there are $M \geq 1$ and $\omega > 0$ such that $\|S^*(t)\| = \|S(t)\| \leq Me^{\omega t}$ for $t \in \mathbb{R}^+$. Then, let A_n be the *Yosida approximation* of the infinitesimal generator A of $S(t)$. That is $A_n = nAR_n(A) = n^2R_n(A) - n \in \mathscr{L}(\mathscr{H})$ with $n \in (\omega,\infty)\cap\mathbb{N}$ where $R_n(A) = (n-A)^{-1}$. It is a well-known result that the sequence of uniformly continuous semigroups $S_n(t) = e^{tA_n}$ satisfies $\|S(t)x - S_n(t)x\| \to 0$ as $n \to \infty$ for any $x \in \mathscr{H}$ and uniformly on compact intervals. Since $S^*(t)$ is also a C_0-semigroup and with generator A^*; the Yosida approximation $A_n^* = nA^*R_n(A^*) = n^2R_n(A^*) - n$ is well defined for $n \in (\omega,\infty)\cap\mathbb{N}$ and $\|S^*(t)x - e^{tA_n^*}x\| \to 0$ as $n \to \infty$ for any $x \in \mathscr{H}$ and uniformly on compact intervals. We observe that $(A_n)^* = n^2((n-A)^{-1})^* - n = n^2(n-A^*)^{-1} - n = A_n^*$, which implies that $S_n^*(t) = (e^{tA_n})^* = e^{t(A_n)^*} = e^{tA_n^*}$ and the assertion is proved.

Then, the sequence $\{\Sigma_n(\cdot)\}_{n=1}^{\infty}$ of solutions of

$$\Sigma_n(t) = S_n(t)\Sigma_0 S_n^*(t) + \int_0^t S_n(t-r)\big(BB^* - \Sigma(C^*C)\Sigma\big)(r)S_n^*(t-r)\,\mathrm{d}r,$$

belongs to $\mathscr{C}(\mathbb{R}^+;\mathscr{I}_p)$ and satisfies $\Sigma_n^*(t) = \Sigma_n(t) \geq 0$ for $t \in \mathbb{R}^+$ by Theorem 3.4 (page 15). Without loss of generality, suppose that $\sup_n \|S_n(t)\| \leq M_\tau$ for $t \in [0,\tau]$(where $M_\tau \geq 1$ was chosen such that $\|S(t)\| \leq M_\tau$ for $t \in [0,\tau]$), therefore by Theorem 3.5,

$$\sup_{t\in[0,\tau_0]} \|\Sigma(t) - \Sigma_n(t)\|_p \to 0,$$

as $n \to \infty$, which implies that $\Sigma^*(t) = \Sigma(t) \geq 0$.

Since $\Sigma^*(t) = \Sigma(t) \geq 0$ for $t \in [0,\tau_0]$ and solves the integral Riccati equation in this interval, we observe that for any $\phi \in \mathscr{H}$ and $t \in [0,\tau_0]$

$$\begin{aligned} 0 \leq \langle \phi, \Sigma(t)\phi\rangle &= \langle \phi, S(t)\Sigma_0 S^*(t)\phi\rangle + \\ &\int_0^t \langle \phi, S(t-r)BB^*(r)S^*(t-r)\phi\rangle - \langle \phi, S(t-r)(\Sigma(C^*C)\Sigma)(r)S^*(t-r)\phi\rangle \,\mathrm{d}s \\ &\leq \langle \phi, S(t)\Sigma_0 S^*(t)\phi\rangle + \int_0^t \langle \phi, S(t-s)BB^*(r)S^*(t-s)\phi\rangle \,\mathrm{d}s. \end{aligned}$$

That is

$$0 \leq \langle \phi, \Sigma(t)\phi\rangle \leq \Big\langle \phi,\ \Big(S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)BB^*(r)S^*(t-s)\,\mathrm{d}s\Big)\phi \Big\rangle,$$

and this latter inequality implies (see Proposition A.1) that

$$\begin{aligned}\|\Sigma(\tau_0)\|_p &\leq \|S(\tau_0)\Sigma_0 S^*(\tau_0) + \int_0^{\tau_0} S(\tau_0 - s)BB^*(s)S^*(\tau_0 - s)\,\mathrm{d}s\|_p \\ &\leq M_\tau^2(\|\Sigma_0\|_p + b_\tau) = \beta.\end{aligned}$$

This allows us to repeat the contraction argument on the interval $[\tau_0, 2\tau_0] \subset [0, \tau]$ and then by Theorem 3.5, we have that $\sup_{t\in[0,2\tau_0]} \|(\Sigma - \Sigma_n)(t)\|_p \to 0$ as $n \to \infty$. Hence again $\Sigma^*(t) = \Sigma(t) \geq 0$ on $t \in [\tau_0, 2\tau_0]$ and we can again use the same argument on $[2\tau_0, 3\tau_0]$, $[3\tau_0, 4\tau_0]$,... etc. □

Remark. We know now that

$$\Sigma(t) = S(t)\Sigma_0 S^*(t) + \int_0^t S(t-s)\big(BB^* - \Sigma(C^*C)\Sigma\big)(s)S^*(t-s)\,\mathrm{d}s, \tag{3.15}$$

has a unique solution $\Sigma(\cdot)$ in $\mathscr{C}([0,\tau];\mathscr{I}_p)$ under Theorem 3.6 hypotheses. Suppose that $BB^*(\cdot) \in \mathscr{C}([0,\tau];\mathscr{I}_p)$ and that $C^*C(\cdot) \in \mathscr{C}([0,\tau];\mathscr{L}(\mathscr{H}))$. Let A be the infinitesimal generator of the C_0-semigroup $S(t)$ over the complex separable Hilbert space $\mathscr{H}$. Since $\mathscr{H}$ is reflexive, $S^*(t)$ is a C_0-semigroup with generator A^* (see [51]). Let $x, y \in \mathscr{D}(A^*)$, and then $\Sigma(\cdot)$ satisfies

$$\begin{aligned}\langle \Sigma(t)x, y\rangle = \langle \Sigma_0 S^*(t)x, S^*(t)y\rangle + \\ \int_0^t \big\langle \big(BB^* - \Sigma(C^*C)\Sigma\big)(s)S^*(t-s)x, S^*(t-s)y\big\rangle\,\mathrm{d}s.\end{aligned}$$

Therefore, $t \mapsto \langle \Sigma(t)x, y\rangle$ is differentiable and a simple computation with the Leibniz integral rule (see [9] for a proof when BB^* and C^*C are constant mappings) shows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \Sigma(t)x, y\rangle = \\ \langle A^*y, \Sigma(t)x\rangle + \langle \Sigma(t)y, A^*x\rangle + \langle BB^*(t)x, y\rangle - \langle \Sigma(t)(C^*C)(t)\Sigma(t)x, y\rangle,\end{aligned}$$

with $\langle \Sigma(0)x, y\rangle = \langle \Sigma_0 x, y\rangle$. Therefore, any solution in $\mathscr{C}([0,\tau];\mathscr{I}_p)$ of the integral Riccati equation (3.15) is a weak solution of the differential equation

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A^* + BB^*(t) - \Sigma(t)(C^*C)(t)\Sigma(t)\ , \tag{3.16}$$

with initial condition $\Sigma(0) = \Sigma_0$. Conversely, any weak solution to this equation can be proven to be a mild solution to the integral Riccati equation (3.15) (See [9] for a proof for constant mappings BB^* and C^*C. The extension for $BB^*(\cdot) \in \mathscr{C}([0,\tau];\mathscr{I}_p)$ and $C^*C(\cdot) \in \mathscr{C}([0,\tau];\mathscr{L}(\mathscr{H}))$ is straightforward). Since the unique solution of this latter equation in the space $\mathscr{C}([0,\tau];\mathscr{I}_p)$ is also a mild solution, these two are equivalent. Therefore, under the hypotheses of Theorem 3.6 and when $BB^*(\cdot) \in \mathscr{C}([0,\tau];\mathscr{I}_p)$ and $C^*C(\cdot) \in \mathscr{C}([0,\tau];\mathscr{L}(\mathscr{H}))$, any weak solution to (3.16) is $\mathscr{I}_p$-valued continuous solution of the integral Riccati equation (3.15).

4. An Application to Sensor Placement. Let $\Omega = (0,1)\times(0,1)$ and consider the convection-diffusion process on the time interval $(0,1)$ with a measuring sensor at (x_0, y_0)

$$\begin{aligned}\frac{\partial T}{\partial t} &= \epsilon^2\Delta T + \Big(a_x\frac{\partial T}{\partial x} + a_y\frac{\partial T}{\partial y}\Big) + b(x,y)\eta(t), \\ h(t) &= \int_\Omega K(x-x_0, y-y_0)T(t,x,y)\,\mathrm{d}x\,\mathrm{d}y + \nu(t),\end{aligned}$$

where $T(t,x,y)=0$ if $(x,y)\in\partial\Omega$, η is a real Wiener process and ν is a real Wiener process that is uncorrelated with η, $\epsilon^2=0.01$ and $K(x,y)=e^{-5(x^2+y^2)}$. In abstract form, the system reads

$$\begin{aligned} z' &= Az + B\eta, \\ h &= Cz + \nu, \end{aligned}$$

where $A=\epsilon^2\Delta+(a_x,a_y)\cdot\nabla$ is strongly elliptic of order 2 (see [60] or [51]), with domain $\mathscr{D}(A)=H^2(\Omega)\cap H_0^1(\Omega)$ and A generates a C_0-semigroup over $L^2(\Omega)$. The input map, is defined as $B:\mathbb{R}\to L^2(\Omega)$ and given by $Ba=b(x,y)a$ with adjoint $B^*:L^2(\Omega)\to\mathbb{R}$ defined by $B^*\varphi=\int_\Omega b(x,y)\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y$. Similarly, the output map $C:L^2(\Omega)\to\mathbb{R}$ is given by $C\varphi=\int_\Omega c(x,y)\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y$, where $c(x,y)=K(x-x_0,y-y_0)$ and (x_0,y_0) is the position of the sensor and its adjoint $C^*:\mathbb{R}\to L^2(\Omega)$ is given by $C^*a=ac(x,y)$.

Then, the optimal location of the sensor is associated to find a pair (x_0,y_0) that minimizes

$$J(x_0,y_0)=\int_0^1 \mathrm{Tr}\,(\Sigma_{(x_0,y_0)}(t))\,\mathrm{d}t,$$

where $\Sigma_{(x_0,y_0)}$ solves the following Riccati equation

$$\dot{\Sigma}(t)=A\Sigma(t)+\Sigma(t)A^*+BB^*(t)-\Sigma(t)(C^*C)(t)\Sigma(t), \tag{4.1}$$

with zero initial conditions.

We consider the following Galerkin-type approximation scheme. The orthonormal set of eigenfunctions of the Laplacian Δ in the unit square is given by $\psi_{m,n}(x,y)=2\sin(\pi mx)\sin(\pi ny)$. We order them using only one parameter first according to its associated eigenvalue $\lambda_{m,n}=-\pi^2(m^2+n^2)$. In the case of two functions sharing the same eigenvalue (e.g. $\psi_{1,3}$ and $\psi_{3,1}$), we place the one with the highest m first. Hence, we obtain the sequence $\{\phi_n\}_{n=1}^\infty$ as $\psi_{1,1},\psi_{2,1},\psi_{1,2},\psi_{2,2},\cdots$.

Define P_n as the orthogonal projector onto $\mathrm{span}\{\phi_1,\phi_2,\cdots,\phi_n\}$. It follows that $P_n^*=P_n$ and $P_n^*P_n=P_n\to I$ strongly. Since $\langle\phi_i,\phi_j\rangle=\delta_{ij}$, the matrix representation $[A_n]\in\mathbb{R}^{n\times n}$ of the approximation $A_n:=P_nAP_n$ is given by

$$[A_n]_{ij}=\epsilon^2\left\langle\phi_i,\Delta\phi_j\right\rangle_{L^2(\Omega)}+a_x\langle\phi_i,\frac{\partial}{\partial x}\phi_j\rangle_{L^2(\Omega)}+a_y\langle\phi_i,\frac{\partial}{\partial y}\phi_j\rangle_{L^2(\Omega)},$$

where $[A_n]_{ij}$ is the i row and j column element of $[A_n]$ and can be computed in closed form. The approximant $C_n:=CP_n$ of C, is given by

$$C_n\phi=\int_\Omega c(x,y)(P_n\phi)(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

and its matrix representation is given by

$$[C_n]=\left(\begin{array}{cccc}\int_\Omega c(\mathbf{x})\phi_1(\mathbf{x})\,\mathrm{d}\mathbf{x} & \int_\Omega c(\mathbf{x})\phi_2(\mathbf{x})\,\mathrm{d}\mathbf{x} & \cdots & \int_\Omega c(\mathbf{x})\phi_n(\mathbf{x})\,\mathrm{d}\mathbf{x}\end{array}\right),$$

where $\mathbf{x}=(x,y)$. It is straightforward to observe that $C_n^*C_n=P_nC^*CP_n$. The matrix representation of $(BB^*)_n:=P_nBB^*P_n$ is given by

$$[(BB^*)_n]=\begin{pmatrix} f(1,1) & f(2,1) & \cdots & f(1,n)\\ f(2,1) & f(2,2) & \cdots & f(2,n)\\ \vdots & \vdots & \ddots & \vdots\\ f(n,1) & f(n,2) & \cdots & f(n,n)\end{pmatrix},$$

where the matrix elements are determined by

$$f(i,j) = \Big(\int_\Omega b(x,y)\phi_i(x,y)\, \mathrm{d}x\, \mathrm{d}y \Big) \Big(\int_\Omega b(x,y)\phi_j(x,y)\, \mathrm{d}x\, \mathrm{d}y \Big).$$

It can be proven (see [52]) that $\{S_n(t)\}$, the uniformly continuous semigroups generated by $\{A_n\}$, $F_n = P_n BB^* P_n$ and $G_n = P_n C^* C P_n$ satisfy the conditions of Theorem 3.5 for $p = 1$ and hence $\sup_{t\in[0,a]} \|\Sigma(t) - \Sigma_n(t)\|_1 \to 0$ for any $a > 0$ and where $\Sigma_n(\cdot)$ satisfies the approximated Riccati equation as in Theorem 3.5.

The approximation of the Riccati equation 4.1 is computed by an implicit Euler scheme:

$$\frac{[\Sigma_n^{k+1}] - [\Sigma_n^k]}{h} = [A_n][\Sigma_n^{k+1}] + [\Sigma_n^{k+1}][A_n]^* + [(BB^*)_n] - [\Sigma_n^{k+1}][C_n]^*[C_n][\Sigma_n^{k+1}],$$

so that for each k the problem reduces to compute an algebraic Riccati equation (see [52] for full details).

We consider three different cases of choices of b, a_x and a_y. In all examples we consider a time-step $h = 0.1$ (no significant changes in the position of minimizers is seen by reducing h further) and in all examples 33 eigenfunctions are used, although changes from 20 to 33 eigenfunctions provide less than 1% difference in values of the objective functional . The first case is determined by uniform noise and zero convective term with parameters $b(x,y) = 10$ and $a_x = a_y = 0$. The minimizer in this case is found exactly at the point $(x_0, y_0) = (0.5, 0.5)$ and the value functional $(x_0, y_0) \mapsto J(x_0, y_0)$behavior can be observed in Figures 4.1(a) and 4.1(d).

The second example is a case with non-uniform noise and zero convective term with data $b(x,y) = 10 + 20e^{-10\big((x-0.2)^2+(y-0.2)^2\big)}$ and $a_x = a_y = 0$. The minimizer of the value functional in this case is found at the point $(x_0, y_0) \simeq (0.38, 0.38)$. In Figures 4.1(b) and 4.1(e) the behavior of $(x_0, y_0) \mapsto J(x_0, y_0)$ can be observed and the displacement of the minimizer, with respect to the previous example, towards the point $(0.2, 0.2)$ is clear.

The third example corresponds to a case with uniform noise and non-zero convective term. Here we use $b(x, y, z) = 10$ and $a_x = a_y = 5$. The minimizer in this case is found at the point $(x_0, y_0) \simeq (0.57, 0.57)$. In Figures 4.1(c) and 4.1(f) we observe the functional $(x_0, y_0) \mapsto J(x_0, y_0)$. The minimizer moves, with respect to the first example, in the opposite direction of the flow of the system.

It should be noted that in some cases there exist ways to circumvent the resolution to large scale Riccati equations in the case of the state estimation problem (see for example [25, 26, 27]).

5. Conclusion. In this paper, we provided results that guarantee the existence of Bochner integrable solutions to the Riccati integral equation in infinite dimensional spaces. In §2, we presented new results concerning the smoothing effect achieved by multiplying a general strongly continuous mapping by continuous $\mathscr{I}_p$-valued functions. This smoothing was use to prove the existence of Bochner integrable solutions of the associated Riccati integral equations. In §3 we established existence and and developed approximation results for the integral Riccati equation. The idea made use of the results developed for uniformly continuous semi-groups in Theorem 3.4 and then proving an a-priori approximation scheme in Theorem 3.5. Finally, combining both results with a fixed property in Theorem 3.6 yielded existence. We formulated an optimal sensor placement problem as a distributed parameter optimal control problem

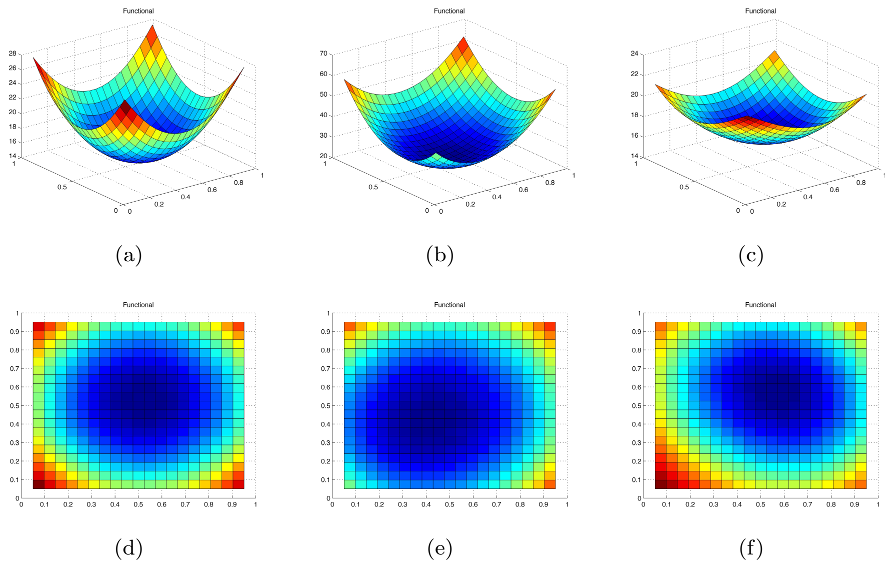

FIG. 4.1. *Value of* $J(x,y) = \int_0^1 \mathrm{Tr}\,(\Sigma_{(x_0,y_0)}(t))\,\mathrm{d}t$ *in two different perspectives for several choices of* b, a_x *and* a_y. *Figures 4.1(a) and 4.1(d) correspond to* $b = 10$ *and* $a_x = a_y = 0$. *Figures 4.1(b) and 4.1(e) depict the case when* $b(x,y) = 10 + 40e^{-5\left((x-0.1)^2+(y-0.1)^2\right)}$ *and* $a_x = a_y = 0$. *Figures 4.1(c) and 4.1(f) correspond to* $b = 10$ *and* $a_x = a_y = 5$.

with the Riccati integral equation as a state constraint. We concluded with with an numerical example to illustrate the theory.

We note that the same ideas can be applied to problems with mobile sensor networks. These results will appear in a future paper.

Appendix A. An inequality on $\mathscr{I}_p$.

PROPOSITION A.1. *Let $A_1, A_2 \in \mathscr{I}_p$ with $1 \le p \le \infty$ satisfy that $0 \le A_1 \le A_2$. Then, $\|A_1\|_p \le \|A_2\|_p$.*

Proof. If $p = \infty$, then from $0 \le \langle A_1\phi, \phi\rangle \le \langle A_2\phi, \phi\rangle$ it follows that $\|A_1\| \le \|A_2\|$ by taking the sup over all $\phi \in \mathscr{H}$ such that $\|\phi\| = 1$.

It is known (see [37]) if $A \in \mathscr{I}_p$ for $1 \le p < \infty$ and $\{\varphi\}_{j=1}^{\omega}$ is some orthonormal system in $\mathscr{H}$, then

$$\left(\sum_{j=1}^{\omega} |\langle A\varphi_j, \varphi_j\rangle|^p\right)^{1/p} \le \|A\|_p,$$

with equality if and only if $A = \sum_{j=1}^{\omega} \langle A\varphi_j, \varphi_j\rangle\langle\cdot, \varphi_j\rangle\varphi_j$.

Then, since A_1 is compact and self-adjoint (for being non-negative), it can be expanded as $A_1 = \sum_{j=1}^{\omega} \sigma_j\langle\cdot, \phi_j\rangle\phi_j$ where $\{\phi_j\}_{j=1}^{\omega}$ is an orthonormal system of eigenvectors of A_1, with $1 \le \omega \le \infty$ and $\sigma_j = \langle A_1\phi_j, \phi_j\rangle$. Hence, by the above inequality we have

$$\|A_1\|_p = \left(\sum_{j=1}^{\omega} |\langle A_1\phi_j, \phi_j\rangle|^p\right)^{1/p} \le \left(\sum_{j=1}^{\omega} |\langle A_2\phi_j, \phi_j\rangle|^p\right)^{1/p} \le \|A_2\|_p,$$

where we have used that $\langle A_1\phi_j, \phi_j\rangle \le \langle A_2\phi_j, \phi_j\rangle$ for all j. $\square$

Appendix B. Proof of Lemma 3.3.

Proof. Step 1: $D(\cdot)$ and $E(\cdot)$ are constant. Suppose first that $D(t) = \Sigma_0 \in \mathscr{I}_p$ and $E(t) = E_0 \in \mathscr{L}(\mathscr{H})$ for all $t \in \mathbb{R}^+$ and Σ_0 and E_0 are non-negative, then we prove that the unique solution to (3.4) is given by $\Sigma(t) = \Sigma_0(I + tE_0\Sigma_0)^{-1} = (I + t\Sigma_0E_0)^{-1}\Sigma_0$. Note that since $\Sigma_0, E_0 \ge 0$, then the spectra of Σ_0E_0 and $E_0\Sigma_0$ are non-negative, i.e., $\sigma(E_0\Sigma_0), \sigma(\Sigma_0E_0) \subset [0,\infty)$ (see [40]). This implies that $-R_{-1}(tE_0\Sigma_0) = (I + tE_0\Sigma_0)^{-1}$ (and $-R_{-1}(t\Sigma_0E_0) = (I + t\Sigma_0E_0)^{-1}$) is well defined where $R_\lambda(A) := (\lambda I - A)^{-1}$ is the resolvent of $A \in \mathscr{L}(\mathscr{H})$ defined for $\lambda \in \rho(A)$, the resolvent set. It follows directly that the adjoint of $(I + t\Sigma_0E_0)^{-1}\Sigma_0$ is given by $\Sigma_0(I + tE_0\Sigma_0)^{-1}$ and direct calculation shows that $(I + t\Sigma_0E_0)^{-1}\Sigma_0 \ge 0$ for all $t \in \mathbb{R}^+$, so that $\Sigma_0(I + tE_0\Sigma_0)^{-1} = (I + t\Sigma_0E_0)^{-1}\Sigma_0$ and hence, $\Sigma(\cdot)$ is well-defined.

If $\lambda \in \rho(A)$, then it is known that $\|R_\lambda(A)\| \le 1/\mathrm{dist}(\lambda, \sigma(A))$ so it follows that $\|(I + tE_0\Sigma_0)^{-1}\| \le 1$ which implies that $\|\Sigma(t)\|_p \le \|\Sigma_0\|_p$ (i.e., the bound in (3.5) holds). By the continuous functional calculus we have that $t \mapsto (I + tE_0\Sigma_0)^{-1} \in \mathscr{L}(\mathscr{H})$ is norm-continuos so that $\Sigma(\cdot) \in \mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$ by application of Proposition 2.1. Furthermore, the identity

$$(I + sE_0\Sigma_0)^{-1} - (I + tE_0\Sigma_0)^{-1} = -(s - t)(I + tE_0\Sigma_0)^{-1}E_0\Sigma_0(I + sE_0\Sigma_0)^{-1},$$

implies that $\frac{\mathrm{d}}{\mathrm{d}t}(I + tE_0\Sigma_0)^{-1} = -(I + tE_0\Sigma_0)^{-1}E_0\Sigma_0(I + tE_0\Sigma_0)^{-1}$ where "$\frac{\mathrm{d}}{\mathrm{d}t}$" is understood in the operator norm-sense, and hence $\frac{\mathrm{d}}{\mathrm{d}t}\Sigma(t) = -\Sigma(t)E_0\Sigma(t)$, i.e., $\Sigma_0(I + tE_0\Sigma_0)^{-1}$ solves (3.4). Uniqueness in $\mathscr{C}(\mathbb{R}^+; \mathscr{L}(\mathscr{H}))$ follows by assuming there is another solution $\tilde{\Sigma}(\cdot)$, then taking the difference if both equations to obtain

$$\|\Sigma(t) - \tilde{\Sigma}(t)\| \le \Big(\sup_{t\in[0,\tau]} \|\Sigma(t)\| + \sup_{t\in[0,\tau]} \|\tilde{\Sigma}(t)\|\Big)\|E_0\| \int_0^t \|\Sigma(s) - \tilde{\Sigma}(s)\| \,\mathrm{d}s,$$

for any $\tau \le t$. An application of a Grönwall's inequality shows uniqueness in the interval $[0,\tau]$ and since $\tau > 0$ is arbitrary, uniqueness of (3.4) in $\mathscr{C}(\mathbb{R}^+; \mathscr{L}(\mathscr{H}))$ follows.

Step 2: $D(\cdot)$ and $E(\cdot)$ are step functions. Suppose now that $E(\cdot)$ is a point-wise non-negative, step $\mathscr{L}(\mathscr{H})$-valued function and $D(\cdot)$ is a monotonically increasing, right-continuous, step $\mathscr{I}_p$-valued function which is also point-wise non-negative. It follows then that $\cup_{n=1}^N I_n = [0,\infty)$, and $I_n = [t_{n-1}, t_n)$ for $n = 1, 2, \ldots, N-1$ and $I_N = [t_{N-1}, \infty)$, and that $D(t) = \sum_{n=1}^N F_n\chi_{I_n}(t)$ and $E(t) = \sum_{n=1}^N G_n\chi_{I_n}(t)$, where $G_n^* = G_n \ge 0$, and $F_n^* = F_n \ge 0$ with $D_1 \le D_2 \le \cdots \le D_N$. Then, we prove that, by the application of the argument in the first paragraph, in each I_n there is a map $\Sigma : \mathbb{R}^+ \to \mathscr{I}_p$ such that its restrictions to each I_n are $\mathscr{I}_p$-continuous (but it is not necessarily continuous on the entire $\mathbb{R}^+$), that solves (3.4) and from the bounds $\sup_{t\in I_n} \|\Sigma(t)\|_p \le \|F_n\|_p$, the bound (3.5) holds. In fact, in the first interval I_1 the application is direct and hence $\Sigma(t) \ge 0$ on $t \in I_1 = [0, t_1)$ and $\sup_{t\in I_1} \|\Sigma(t)\|_p \le \|D_1\|_p$; for $t \in I_n$ with $n = 2, 3, \ldots, N$, $\Sigma(\cdot)$ satisfies

$$\Sigma(t) = \Sigma(t_{n-1}^-) - \int_{t_{n-1}}^t (\Sigma E\Sigma^*)(s) \,\mathrm{d}s,$$

If $\Sigma(t_{n-1}^-) \ge 0$ and $\sup_{t\in I_{n-1}} \|\Sigma(t)\|_p \le \|D_{n-1}\|_p$, then by step 1, $\Sigma(\cdot)$ is well defined on I_n and $\sup_{t\in I_n} \|\Sigma(t)\|_p \le \|\Sigma(t_{n-1}^-)\|_p \le \|D_{n-1}\|_p \le \|D_n\|_p$. Therefore, the proof follows by induction. Uniqueness follows by the same argument as in step 1 on each interval I_n.

Step 3: General $D(\cdot)$ and $E(\cdot)$. If $E(\cdot) \in L^{\infty}_{loc}(\mathbb{R}^+; \mathscr{L}(\mathscr{H}))$, $D(\cdot) \in \mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$, both are point-wise non-negative and $t \mapsto D(t)$ is monotonically increasing. Then, there are step functions of the type in the above paragraph F_n and G_n, such that

$$a_n(\tau) := \sup_{t\in[0,\tau]} \|D(t) - F_n(t)\|_p \to 0 \quad \text{and} \quad b_n(\tau) := \int_0^{\tau} \|E(s) - G_n(s)\| \,\mathrm{d}s \to 0,$$

for any $\tau > 0$. We consider the sequence $\{\Sigma_n\}$ of pointwise non-negative mappings solutions to

$$\Sigma_n(t) = F_n(t) - \int_0^t \Sigma_n(s) G_n(s) \Sigma_n^*(s) \,\mathrm{d}s, \tag{B.1}$$

so that $\sup_{t\in[0,\tau]} \|\Sigma_n(t)\|_p \leq \|F_n(\tau)\|_p \leq M_1(\tau)$ and also $\int_0^{\tau} \|G_n(s)\| \,\mathrm{d}s \leq M_2(\tau)$ where $M_1(\tau), M_2(\tau)$ do not depend on $n \in \mathbb{N}$. From (B.1), it can be proven that

$$\|(\Sigma_m - \Sigma_n)(t)\|_p \leq$$
$$a_n(\tau) + M_1(\tau)^2 b_n(\tau) + \int_0^t M_1(\tau)^2 \Big(\|E_m(s)\| + \|G_n(s)\|\Big) \|(\Sigma_n - \Sigma_m)(s)\|_p \,\mathrm{d}s,$$

and it follows by an application of Grönwall's inequality that for each t, $\{\Sigma_n(t)\}$ is a Cauchy sequence in $\mathscr{I}_p$. We denote $t \mapsto \Sigma(t)$ to the map of this limits which is Bochner integrable for being the pointwise limit of Bochner measurable functions. Furthermore $\sup_{t\in[0,\tau]} \|(\Sigma_m - \Sigma_n)(t)\|_p \to 0$ uniformly in m, n which implies that $\sup_{t\in[0,\tau]} \|(\Sigma - \Sigma_n)(t)\|_p \to 0$ and hence for each t, $\int_0^t \Sigma_n(s) G_n(s) \Sigma_n^*(s) \,\mathrm{d}s \to \int_0^t \Sigma(s) E(s) \Sigma^*(s) \,\mathrm{d}s$ in $\mathscr{I}_p$ follows since in addition $b_n(t) \to 0$ as $n \to \infty$. It follows that $\Sigma(\cdot)$ solves (3.4) and satisfies the bound (3.5). Since $\Sigma(\cdot) \in L^2_{loc}(I; \mathscr{I}_p) \subset L^2_{loc}(I; \mathscr{I}_{2p})$, then by Theorem (3.1), $\Sigma(\cdot) = \gamma(\Sigma)(\cdot) \in \mathscr{C}(\mathbb{R}^+; \mathscr{I}_p)$. Uniqueness follow by the assuming that there is another solution $\tilde{\Sigma}(\cdot)$ and an application of Grönwall's inequality. □

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