Synchronization and Coherence of Dynamical Systems: Networks of Coupled Rössler Attractors

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Abstract

In order to quantify the onset of phase and full synchronization in networks of heterogeneous oscillators, we propose two order parameters. We then use these order parameters to explore the onset of synchronization in various networks of nonidentical Rössler attractors. We present our main findings, including: monotonic and non-monotonic phase and full synchronization, as results of anti-synchronization and "network frustration"; (partial) phase synchronization in subgroups of networks; and gradual full synchronization in subgroups of networks and overall.

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1 Introduction and Background

> In the late twentieth century, while studying weather models, mathematician Ed Lorenz was surprised to see that: when he changed his model's inputs only slightly, the model gave him not slightly, but drastically, different results. Lorenz had, in fact, encountered *a chaotic system*.

In the study of dynamical systems, e.g. maps and differential equations, chaos refers to one of three general types of behavior; the other two are fixed points and periodic orbits. From these categories, we can identify one of the many interesting characteristics of chaos: trajectories never repeat themselves. Another characteristic, as Lorenz discovered, is that chaotic systems are extremely sensitive to initial conditions. Intrigued by these characteristics and more, mathematicians have researched chaos in an attempt to understand it further.

In addition to in weather models, chaos manifests itself in many other places in nature, including: electrical circuits, lasers, and solar satellites [1]. There also exist several primarily mathematical systems, e.g. the logistic map, that exhibit such behavior.

1.1 The Rössler attractor

The Rössler attractor is one well-known example of chaos. Designed in 1976 by Otto Rössler, it was originally intended to "model equilibrium in chemical reactions" [2]. In most contexts, though, it is more of a specimen for chaos studies.

The Rössler attractor is defined by the following three dimensional set of nonlinear, ordinary differential equations

$$\dot{x} = -\omega y - z \tag{1}$$

$$\dot{y} = \omega x + ay \tag{2}$$

$$\dot{z} = f + z(x - c) \tag{3}$$

where ω, a, f, and c are constant parameters. It is important to note that only for certain ranges of these parameters is the behavior chaotic. In this paper, we will consider only small variations of one set of parameters

$$a = 0.165 \qquad f = 0.2 \qquad c = 10 \qquad \omega = 0.97$$

which are known to provide chaotic trajectories.

1.2 Synchronization and Coherence

In the study of chaos, one interesting phenomenon called *synchronization* [3, 4, 5] crops up when two or more chaotic systems are coupled (i.e. they interact with each other). It refers to when systems are coupled in such a way that they act "in unison" with each other (while the chaotic behavior of the individual systems may remain).

There are many types of synchronization, but in this paper we will limit our discussion to: *coherence*, *phase synchronization* and *full synchronization*.

Phase synchronization refers to when all trajectories are locked in phase (for an extended period of time). For example, we see in Figure (1) that the blue and green trajectories are in phase but differ in amplitude.

Full synchronization refers to when all trajectories are locked in phase and amplitude (for an extended period of time). In Figure (2), we see the two trajectories lie directly on top of each other; that is, they agree in both phase and amplitude.

Coherence refers to any type of partial synchronization, e.g. phase synchronization of a subset of oscillators, when the entire network is not synchronized in phase.

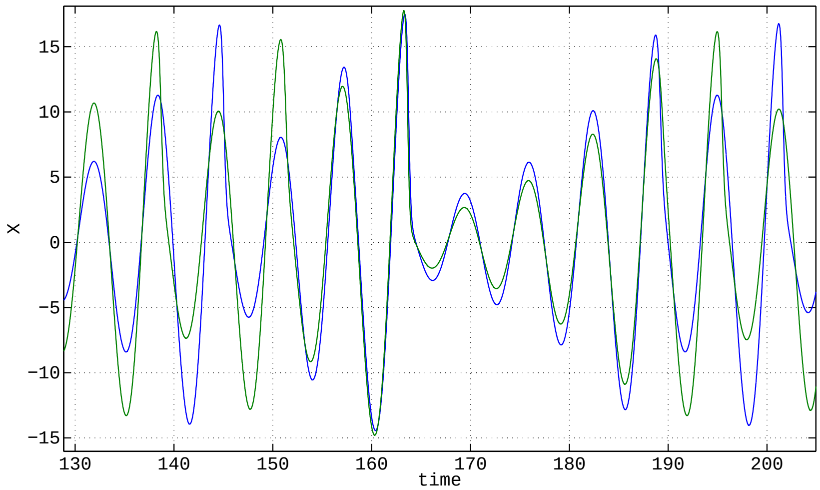

Figure 1: Phase synchronization with two Rösslers

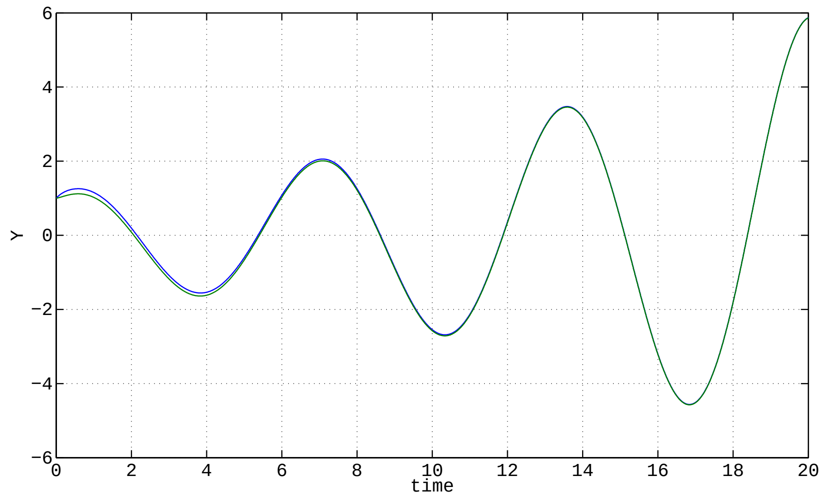

Figure 2: Full synchronization with two Rösslers

We should point out that synchronization is not just a mathematical phenomenon. In neurobiological networks, for example, where "irregular (chaotic) dynamics occur naturally" [6], an understanding of synchronization would be a tremendous breakthrough. Also, in engineering applications that involve chaotic systems, e.g. lasers, it would be useful to understand the mechanisms that lead to the onset of coherence.

1.3 Motivation and Research Questions

Given the widespread appearance of chaos, and the potential implications of understanding synchronization, many have sought to determine how it is that synchronization happens.

Restrepo et al [7] determined that there is a critical coupling strength that leads to the onset of coherence. This critical coupling strength, they showed, depends on the uncoupled dynamics of the oscillators and the adjacency matrix. This analysis, though, is limited by their assumption that each oscillator has a large number of neighbors.

Ott et al [8] also explored synchronization in the case of globally coupled chaotic and periodic oscillators.

The onset of synchronization in small, rather sparse networks remains minimally understood. In this paper we will investigate the onset of phase and full synchronization, as well as partial synchronization (coherence), in sparse networks of heterogeneous Rössler attractors. We are interested, in particular, in exploring the following questions:

1. Qualitatively speaking, how is the transition to phase and full synchronization? Is it abrupt? Does synchronization seem to happen at once? Or is it gradual? Do oscillators synchronize first in subgroups, then in larger groups, and then finally over the entire network?

2. Is the onset of phase and full synchronization monotonic? That is, does a greater coupling strength lead to more synchronization?

2 Research Methods

2.1 N Coupled, Nonidentical Rössler Attractors

In this paper, we will couple N nonidentical Rössler attractors. Let the ith oscillator be defined by

$$\dot{\mathbf{x}}_i = \begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \end{pmatrix} = \mathbf{F}(\mathbf{x}_i) + kK \sum_{j=1}^{N} A_{ij}(\mathbf{x}_j - \mathbf{x}_i) \tag{4}$$

where: $\mathbf{F}(\mathbf{x}_i)$ is the set of ordinary differential equations that define a single Rössler attractor (Eqs. (1), (2), and (3)); A is the $(N \times N)$ adjacency matrix (i.e. $A_{ij} = 1$ if the jth Rössler drives the ith Rössler, $A_{ij} = 0$ otherwise); k is the (positive) scalar coupling strength; and K is the (3×3) diagonal coupling matrix (e.g. if K is the identity matrix, then all three components of the Rösslers are coupled).

2.2 Order Parameters

We have seen that in the case of two coupled Rösslers, synchronization is relatively easy to identify; in each component, we need only look at two curves and determine if they agree in phase and possibly also in amplitude. However, when we scale up just even to ten Rösslers, for example, the task of identifying synchronization (and particularly its onset) is much more difficult. To identify phase synchronization, one must determine if all components of the ten trajectories are locked in phase; and the more challenging part is to say when this first happens. For full synchronization, it is a bit easier as one must only look to see if all trajectories are identical. We shall note, though, that there are other interesting possibilities—namely, partial synchronization—that are of interest, yet difficult to identify with the eye. Clearly, inspection is not a good enough method for identifying all of these interesting phenomena.

We propose, instead, two different "order parameters" that quantify the onset of synchronization in its different forms: phase synchronization, full synchronization, and partial forms of synchronization in between. In each case, the idea is to define a function, $o(t)$, that takes certain values for synchronization (i.e. either full or phase) and others for asynchronization; in between, the values indicate some type of partial synchronization. Since it is a function of time, we will look at the root mean square average, i.e. o_{rms}, over a window of time, i.e. $\tau \leq t \leq \tau + T$, where the oscillator has relaxed onto the attractor.

$$O = o_{rms} = \sqrt{\frac{\int_{\tau}^{\tau+T} [o(t)]^2 \, dt}{T}}$$

This scalar quantity O will therefore indicate the coherence (or incoherence) of the network.

2.3 Full order parameter, $\mathcal{R}$

The case of full synchronization is defined to be when all N oscillators in the network are identical for an extended period of time. That is,

$$\mathbf{x}_i(t) - \mathbf{x}_j(t) \equiv 0 \qquad \text{for} \qquad 1 \leq i, j \leq N, \quad \tau_0 \leq t \leq \tau_1, \ \tau_0 \ll \tau_1$$

Thus, for all t in the window, the quantity

$$\sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} \left\| \mathbf{x}_j(t) - \mathbf{x}_i(t) \right\|$$

is (close to) zero in the case of full synchronization and large in the case of asynchronization. We have thus defined a quantity that tells us when full synchronization occurs, so we will now only scale this quantity by dividing by two factors

$$\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij} \tag{5}$$

$$\mathbf{x}^{\star}_{rms} \tag{6}$$

where (5) is the sum over the adjacency matrix, counting the number of connections in the network, and (6) is the root mean square average of the chaotic oscillations of a representative oscillator, that is,

$$\mathbf{x}^{\star}_{rms} = \sqrt{\frac{\displaystyle\int_{\tau}^{\tau+T} [x^{\star}(t)]^2 + [y^{\star}(t)]^2 + [z^{\star}(t)]^2 \; dt}{T}}.$$

By dividing by (5), the quantity does not depend on the size of the network (i.e. the number of ones in the adjacency matrix), and by dividing by (6), it does not depend on the magnitude/units of the oscillations (i.e. it is dimensionless).

Let us then define

$$r(t) := \frac{\displaystyle\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij} \left\|\mathbf{x}_j(t) - \mathbf{x}_i(t)\right\|}{\displaystyle\mathbf{x}^{\star}_{rms}\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}}. \tag{7}$$

We will now only consider the root mean square average of $r(t)$ over a window of time after

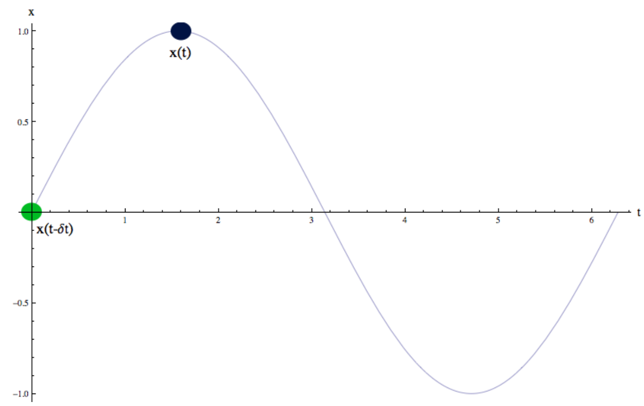

Figure 3: A sine curve, two points $x(t)$ and $x(t - \delta t)$

the oscillators have relaxed onto the attractor.

$$\mathcal{R} = r_{rms} = \sqrt{\frac{\int_{\tau}^{\tau+T} [r(t)]^2 \, dt}{T}} \tag{8}$$

We will call this scalar quantity $\mathcal{R}$ the full synchronization order parameter, where $\mathcal{R} \approx 0$ implies full synchronization and $\mathcal{R} \approx 1$ implies asynchronization.

2.4 Phase order parameter, $\mathcal{P}$

The case of phase synchronization is defined to be when all N oscillators in the network are locked in phase for an extended period of time. To quantify this behavior, we will rely on the oscillatory behavior of the x-component of the Rössler system. (Note: The y-component is also oscillatory, so the same analysis could be done with that component; the z-component, however, is not oscillatory, so it would not make a good choice.)

For the sake of illustration, let us assume for the moment that the x-component of the

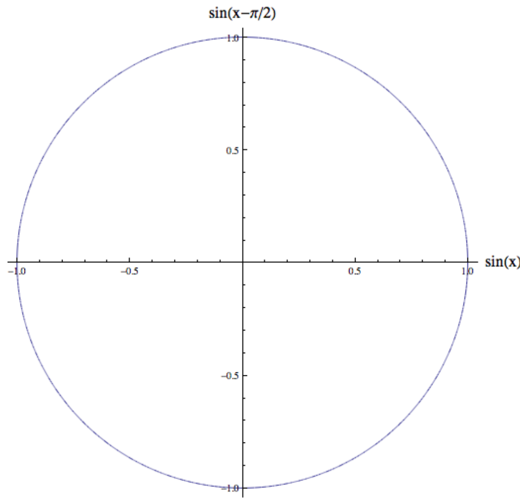

Figure 4: A phase diagram of $\sin(t)$ v.s. $\sin(t-\delta t)$

Rössler is a sine curve:

$$x(t) = \sin(t).$$

Consider two points, $x(t)$ and $x(t-\delta t)$, where δt is one-fourth of an oscillation (i.e. $\delta t = \pi/2$ in our case). [See Figure (3).] Moreover, let us consider the complex number,

$$x(t) + ix(t-\delta t).$$

We can generate a phase diagram by plotting that number in the complex plane as t increases. [See Figure (4).] The angle in the complex plane as we move around the circle[1] is now a phase angle for the oscillator.

[1] The circle is a result of a constant amplitude for the sine curve; in our chaotic systems, this may become more elliptical because of varying amplitudes, but the normalization step will make this a non-issue.

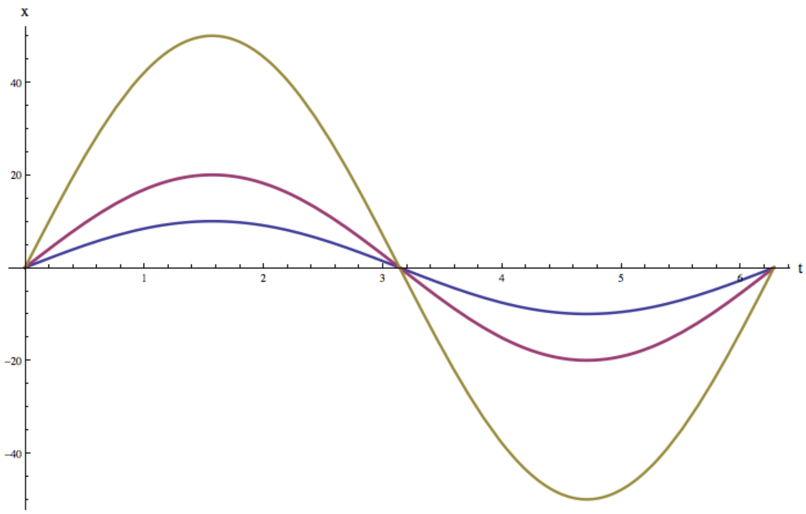

Figure 5: Multiple sine curves synchronized in phase

Let us extend our analysis now to multiple sine curves that are synchronized in phase

$$x_j(t) = A_j \sin(t),\ j = 1, 2, 3.$$

[See Figure (5).] Without loss of generality, we will look at $t = \pi/2$. We see that the complex numbers represented by each of these curves at $t = \pi/2$ are:

$$A_1 \sin(t = \pi/2) + iA_1 \sin(t - \delta t = 0) = A_1 + i0$$

$$A_2 \sin(t = \pi/2) + iA_2 \sin(t - \delta t = 0) = A_2 + i0$$

$$A_3 \sin(t = \pi/2) + iA_3 \sin(t - \delta t = 0) = A_3 + i0$$

All have the same angle in the complex plane (i.e. 0), but different magnitudes (i.e. A_j). Then, by normalizing these complex numbers (i.e. placing them on the unit circle), all of the curves at $t = \pi/2$ would be represented by the same complex number: $1 + i0$, with phase angle 0 and magnitude 1. Since these curves are synchronized, the same holds for

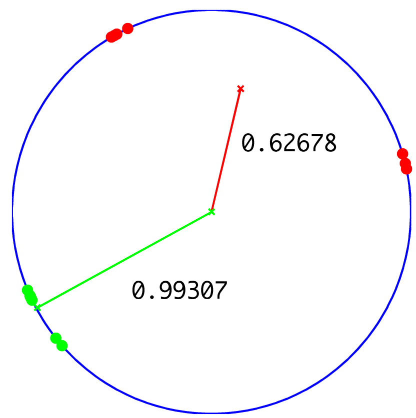

Figure 6: The red trajectories are not phase synchronized, so the norm of their average is approximately 0.65. The green trajectories are phase synchronized, so the norm of their average is approximately 0.99.

all t.

In general: if trajectories are synchronized in phase, then (when normalized) they will move together on the unit circle (i.e. same phase angle); however, if they are not synchronized, then (when normalized) they will move separately around the unit circle (i.e. different phase angles). To measure phase synchronization, then, we can average these complex numbers over all oscillators. This average, which is itself a complex number, will have a magnitude close to 1 if the curves are synchronized in phase. Otherwise, if the complex numbers are distributed over the entire unit circle, then this average will have a much smaller magnitude.

Therefore let us define

$$p(t) = \frac{1}{N} \left\| \sum_{j=1}^{N} \frac{x_j(t) + i x_j(t - \delta t)}{\| x_j(t) + i x_j(t - \delta t) \|} \right\|. \tag{9}$$

[See Figure (6) for an example of how $p(t)$ works.] As we did with $r(t)$, we will consider the root mean square average of $p(t)$ over a window of time after the oscillators have relaxed onto the attractor.

$$\mathcal{P} = p_{rms} = \sqrt{\frac{\displaystyle\int_{\tau}^{\tau+T} [p(t)]^2 \, dt}{T}} \tag{10}$$

We will call this scalar quantity $\mathcal{P}$ the phase synchronization order parameter, where $\mathcal{P} \approx 1$ implies phase synchronization and $\mathcal{P} \approx 0$ implies asynchronization.

3 Results and Discussion

In all of the following investigations, we will use:

- Random initial conditions: The initial conditions for all components of all Rössler attractors will be randomly generated between 0.8 and 1.2. By randomly generating them, instead of choosing them to be identical, we can avoid the problem of trajectories initially appearing synchronized. Also, we know that these initial conditions lie within the basins of attraction of the Rössler attractors for the chosen parameter values.

- RMS for $\mathcal{R}$: We will use $x^{\star}_{rms} = 10.7177$, where this has been computed as described in Section (2.3) using $a = 0.165$, $f = 0.2$, $c = 10$, and $\omega = 0.97$.

The values of the other parameters will be discussed as needed.

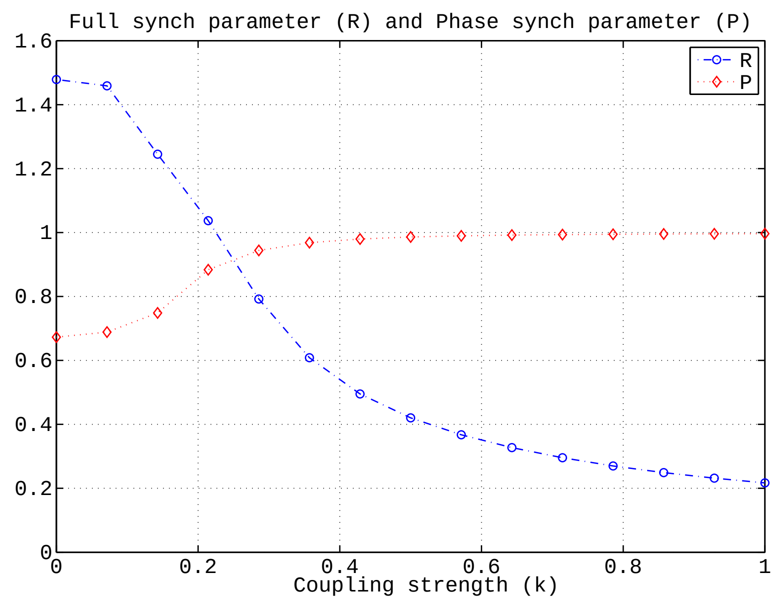

Figure 7: Two Rösslers phase and fully synchronize monotonically

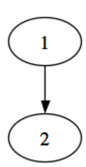

Figure 8: Two Rösslers

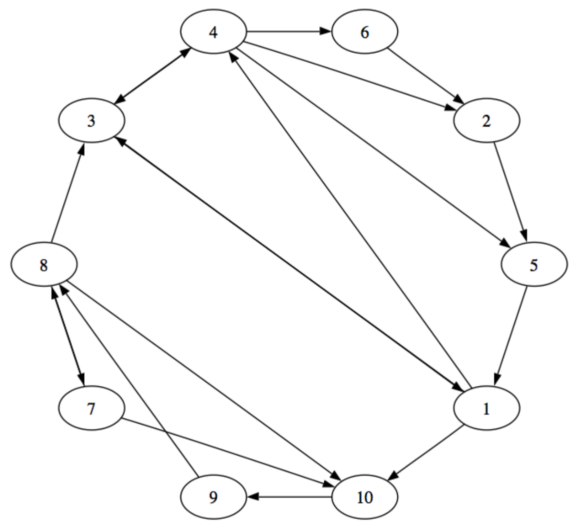

Figure 9: Ten Rösslers, adjacency matrix with two subgroups

Figure 10: Simulation information

	Two Rösslers	Ten Rösslers, Subgroups
A	See Figure (8)	See Figure (9)
a	all 0.165	all 0.165
f	all 0.2	all 0.2
c	(1) 9 (1) 11	(6) 9 and $\left[\begin{array}{cccc} 6 & 6.1 & 10.9 & 11 \end{array}\right]$
ω	(1) 0.9 (1) 1.1	(6) 0.95 and $\left[\begin{array}{cccc} 0.91 & 0.92 & 1.04 & 1.05 \end{array}\right]$
K	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

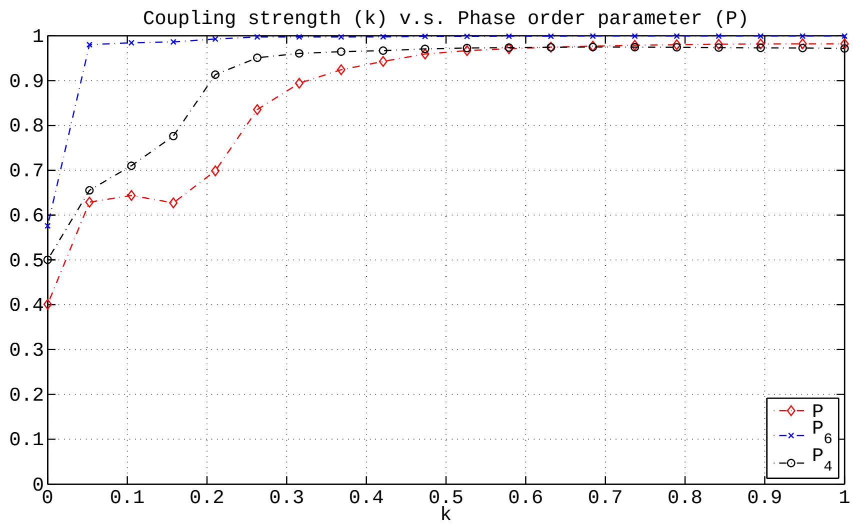

Figure 11: Phase synchronization, in both subgroups and overall

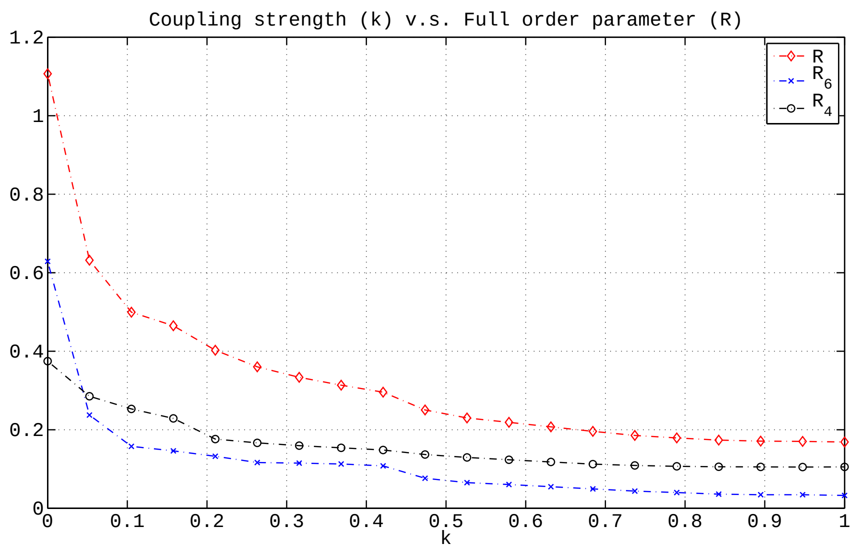

Figure 12: Full synchronization, in both subgroups and overall

3.1 Questions: Does phase synchronization happen all at once? Does full synchronization happen all at once?

No. Neither phase synchronization nor full synchronization happens all at once. In fact, both can happen gradually, or first in subgroups.

In the simple case of two coupled Rössler attractors [see Figure (10) for more information], we find as in Figure (7) that neither phase synchronization nor full synchronization happens all at once. In this case, both happen gradually as the coupling strength is increased.

With the suspicion that synchronization is not instant in more complicated networks, next we design a network of ten Rösslers that contains two subgroups: one of six identical attractors, the other of four nonidentical attractors. There are two connections between the networks. [See Figure (10) for more information.]

We find, as seen in Figure (11), that phase synchronization happens as one would expect in a network with subgroups: first in the subgroups (and quicker in the subgroup of six with more internal connections and more similar oscillators), then in the overall network. One should also note that: (i) in the subgroup of six, $\mathcal{P}$ reaches 1 (unlike in the subgroup of four and overall) because the oscillators in this subgroup are identical and can therefore synchronize exactly in phase; and (ii) the overall network is eventually more synchronized in phase than the subgroup of four, which can be understood by the similarity between two oscillators in that group and the subgroup of six, therefore allowing eight oscillators to synchronize (almost perfectly) in phase. This suggests that phase and full synchronization is limited by the similarity between the oscillators.

As seen in Figure (12), we find that full synchronization happens gradually (both in the subgroups and overall). It also seems that the rate of full synchronization is greatest in the identical subgroup and in the overall network (with six identical oscillators, and two similar

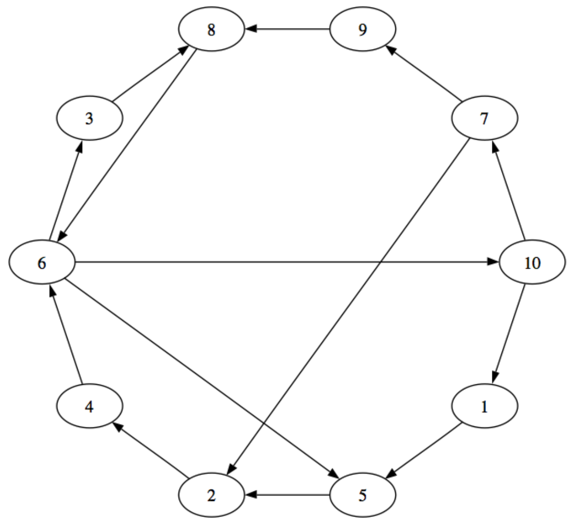

Figure 13: Ten Rösslers, sparse adjacency matrix

to that group). This suggests that oscillator similarity affects the rate of synchronization.

3.2 Questions: Is phase synchronization monotonic? Is full synchronization monotonic?

Not necessarily. We find for some networks that phase and full synchronization is monotonic; for other networks, anti-synchronization and "network frustration" lead to non-monotonic phase and full synchronization.

In the same case of two Rösslers that we investigated before, we see in Figure (7) that both phase and full synchronization are monotonic. As the coupling strength, k, increases, the phase order parameter $\mathcal{P}$ increases toward 1 and the full order parameter $\mathcal{R}$ decreases toward 0. The monotonicity may be related to the simple interactions (i.e. only one connection, two Rösslers) in this network. [See Figure (10) for more information on the simulation.]

In the case of ten Rösslers, we find two behaviors which lead to non-monotonic syn-

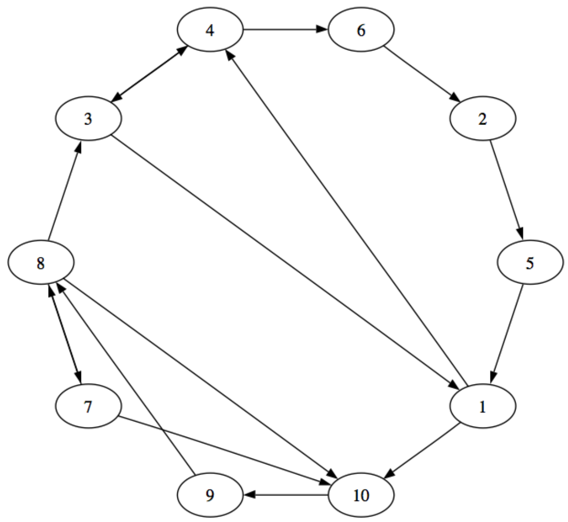

Figure 14: Ten Rösslers, sparse adjacency matrix with two subgroups

Figure 15: Simulation information

	10 Rösslers, Sparse	10 Rösslers, Sparse Subgroups
A	See Figure (13)	See Figure (14)
a	all 0.165	all 0.165
f	all 0.2	all 0.2
c	(5) 9 (5) 11	(6) 9 (4) 11
ω	(5) 0.95 (5) 1.05	(6) 0.95 (4) 1.05
K	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

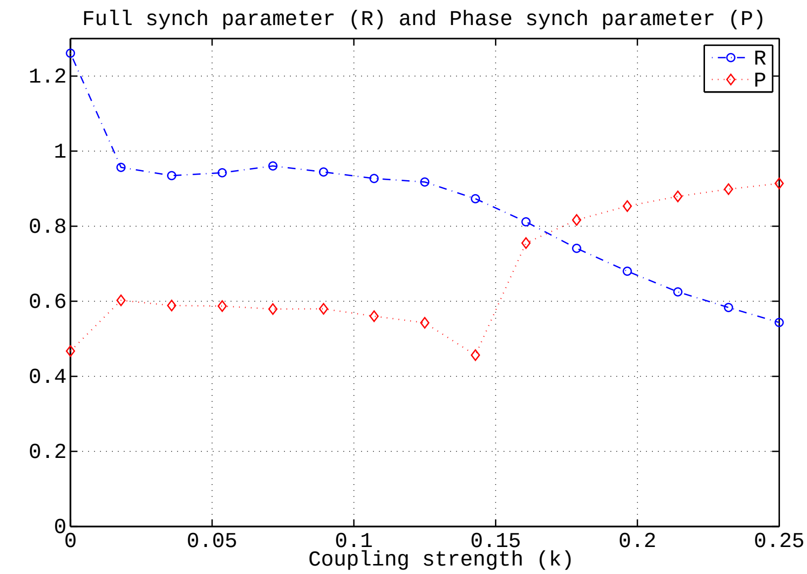

Figure 16: Anti-synchronization leads to non-monotonic synchronization

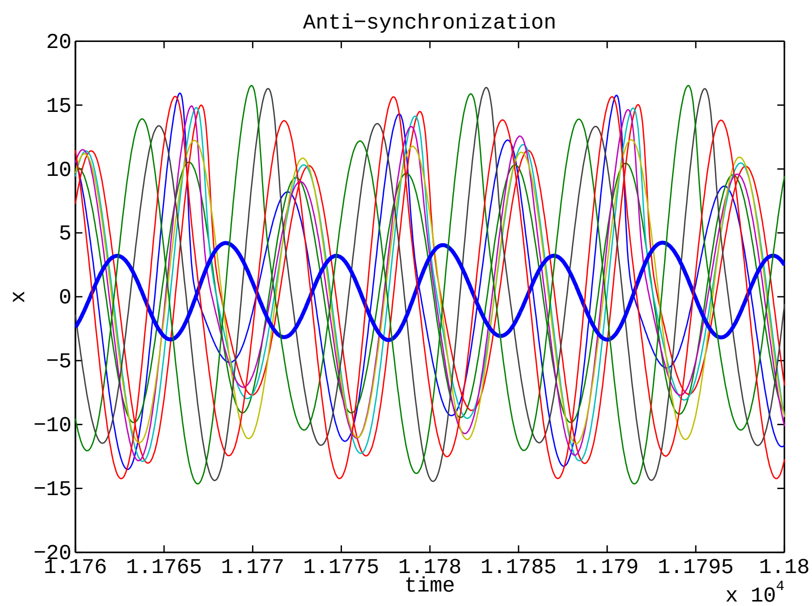

Figure 17: Anti-synchronization of one of the Rösslers (bold blue trajectory)

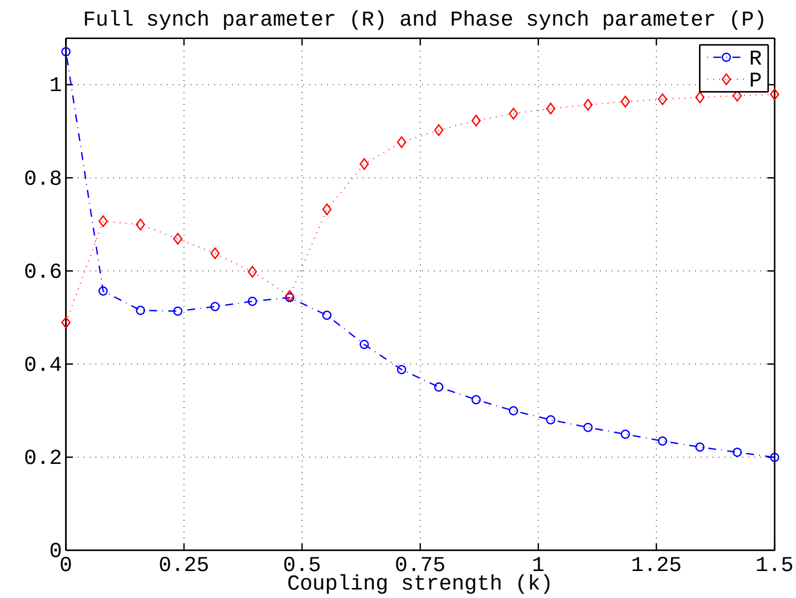

Figure 18: Network frustration leads to non-monotonic synchronization

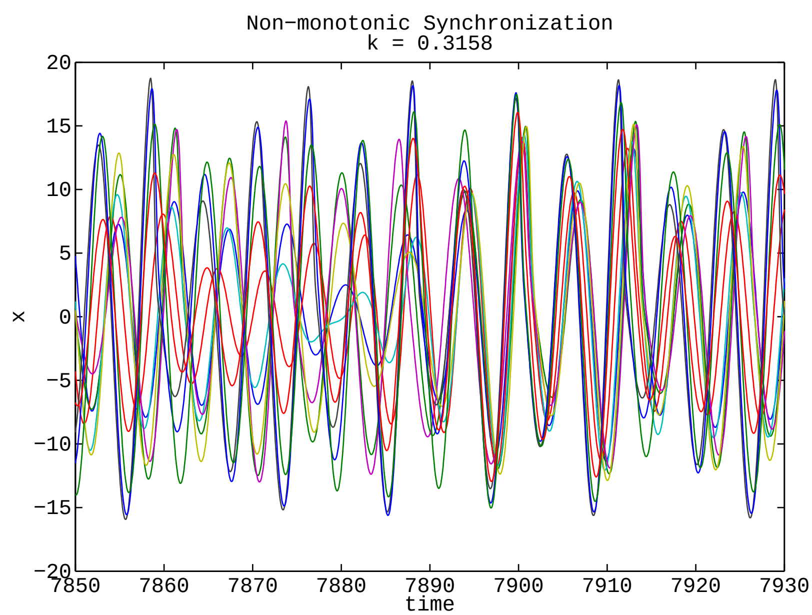

Figure 19: Trajectories with network frustration

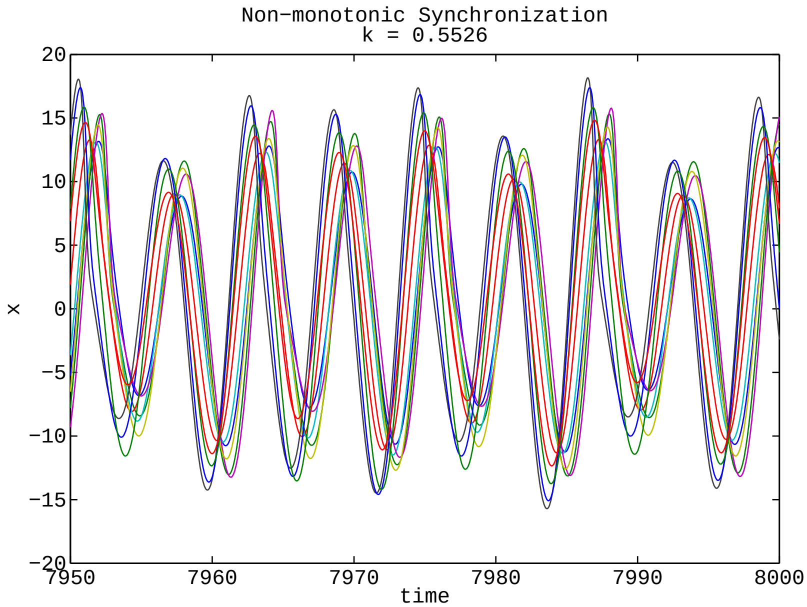

Figure 20: Trajectories without network frustration

chronization in phase and overall: (partial) anti-synchronization and "network frustration." [See Figure (15) for more information on the simulations. Anti-synchronization occurs with the sparse adjacency matrix; network frustration occurs with the sparse subgroup matrix.]

Anti-synchronization As seen in Figure (16), for coupling strengths, k, between 0.05 and 0.14, the phase order parameter $\mathcal{P}$ decreases. This happens because one of the Rössler attractors anti-synchronizes with the rest of the network.[2] [See Figure (17).] It is surprising to see anti-synchronization with Rössler attractors coupled in this way. We are unsure of the mechanism that leads to this phenomenon. $\mathcal{R}$ also increases slightly between 0.05 and 0.06, but this increase is not attributable to any distinct behavior or phenomenon.

Network frustration As seen in Figure (18), for coupling strengths, k, between 0.1 and

[2]When computing $\mathcal{P}$, anti-synchronization leads to points that are opposite each other on the unit circle; then, when you add these points as complex numbers, the average has a smaller norm.

0.45, the phase order parameter $\mathcal{P}$ decreases; also, for k between 0.25 and 0.45, the full order parameter $\mathcal{R}$ increases. This can be explained by the behavior seen for $k = 0.3158$ in the interval $7850 \leq t \leq 7885$ in Figure (19), which we are calling network frustration. Although we are not sure of the exact mechanism causing this behavior, it seems that the Rössler attractors (through interacting with each other) are experiencing cycles of: being driven out of agreement in amplitude and in phase, and then returning to coherence. At $t = 7875$, the Rösslers are essentially in one of two phases; the oscillators in one group are at their maxima, while those in the other group are at their minima. At this same time ($t = 7875$), there seem to be essentially nine different amplitudes for the ten Rösslers. Then, by $t = 7895$, the Rösslers nearly agree in phase and in amplitude. This pattern repeats itself again and again for this coupling strength. When k increases to 0.5526, as seen in Figure (20), there is no longer any network frustration; thus, $\mathcal{P}$ and $\mathcal{R}$ are higher and lower respectively. As k continues increasing from this point, the network does not experience any more frustration, and the order parameters continue monotonically.

4 Conclusions and Future Work

The proposed order parameters, $\mathcal{R}$ and $\mathcal{P}$, successfully quantify the onset of partial, phase, and full synchronization in networks of Rössler attractors. In general, the phase order parameter could be used with any coupled attractors provided one of the components of the attractors is oscillatory. The full order parameter could also be used in networks of different attractors.

Using the order parameters, a few conclusions were drawn:

Synchronization not all at once. Neither phase nor full synchronization happens at once. In our examples provided, phase synchronization happened both gradually and in subgroups. Full synchronization happened gradually in all of our example networks.

Monotonic and non-monotonic. Synchronization is not necessarily monotonic. Interesting behaviors like anti-synchronization and network frustration can lead to non-monotonic synchronization.

In future work, we would like to investigate:

More behaviors. We would like to continue exploring and observing the onset of synchronization for different networks, including: adjacency matrices of varying sparsity, the limit of nonidentical to identical oscillators, and more.

Analytic connections. We would like to explore analytic connections between network topology, oscillator dynamics and the onset of synchronization in this case where assumptions of many neighbors and globally coupled oscillators do not apply.

Applications. We would like to investigate potential applications of these order parameters. One particular application we have considered is in the detection of similar (and dissimilar) oscillators using coupling and our order parameters.

5 Acknowledgements

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