Numerical Study of a Quantum Memory Model

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July 31, 2009

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Abstract. Quantum computers have the potential to be much more powerful than their classical counterparts. However, there are still many obstacles that need to be overcome before quantum computing can become a viable tool, including problems relating to storing information in quantum memory. One particular quantum memory model-the three-level quantum system-aims to be more manipulable than other current models while also retaining information longer. Since this model can be computationally expensive, this study uses several different methods to simulate (using Matlab) three special cases of this model.

The graphs show that-given enough time-each numerical method is able to simulate the evolution of the quantum system according to theoretical predictions for each specific case. Also, with respect to the analytic solution requiring numerical integration, an error analysis shows the differing accuracy between the different numerical approaches in one of the special cases.

This final report is submitted in partial fulfillment of the
National Science Foundation REU and Department of Defense ASSURE Program
George Mason University
June 1, 2009 - July 31, 2009

1. Introduction

Lov Grover's quantum search algorithm and Peter Shor's quantum integer factorization algorithm are just a couple indicators that demonstrate the potential speedup (quadratic and exponential, respectively) quantum computers may provide over classical computers.[2] This is made possible by the way quantum computers store information: qubits, the quantum counterpart to bits. The qubit model in this report is a three-level quantum system interacting with two lasers. This three-level model provides a couple advantages over a two-level qubit model:

- The two lowest states are only finely separated. As a result, this quantum system retains information much longer than a two-level system whose states are fairly well-separated.
- We can indirectly manipulate the lowest nonground state by using the third state of the system. This allows us to use optic lasers to do the manipulation, which already has a good technological base. Directly manipulating the two lowest states would require energy on the order of microwaves, in which case the technology is far less developed.

Because of these advantages, the three-level system is worthwhile to investigate as a qubit model. However, because of the complex setup of the system, we need to see how well we can simulate this system with different approaches (numerical methods, an exact solution, etc.).

2. Background and Research Methods

2.1. **Problem Setup.** Figure 1 shows a potential quantum system model for a qubit. We use the Schödinger Equation

$$\dot{\rho} = -\frac{\imath}{\hbar}[H, \rho] - \imath R\rho \tag{1}$$

to describe the evolution of the quantum system over time, where H is the system Hamiltonian, ρ is the density matrix, and R is the relaxation operator.

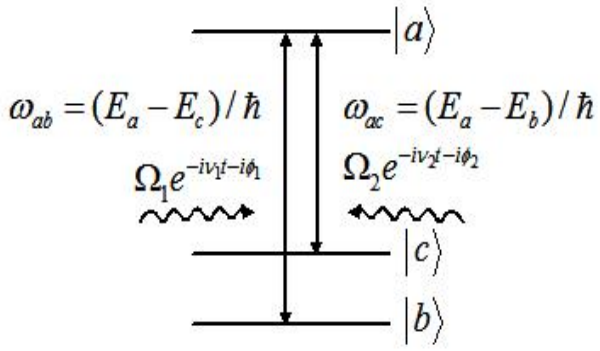

FIGURE 1. Three-level system interacting with two laser fields

The system Hamiltonian H describes the total energy in the quantum system.

$$H = H_{in} + H_0 \tag{2}$$

which is the energy due to the laser pulses and energy already in the quantum system, respectively. So,

$$\begin{aligned} H_{in} = &-\frac{\hbar}{2}\Omega_1 e^{-\imath\phi_1 - \imath\nu_1 t} \mid a\rangle\langle b \mid -\frac{\hbar}{2}\Omega_2 e^{-\imath\phi_2 - \imath\nu_2 t} \mid a\rangle\langle c \mid \\ &-\frac{\hbar}{2}\Omega_1 e^{\imath\phi_1 + \imath\nu_1 t} \mid b\rangle\langle a \mid -\frac{\hbar}{2}\Omega_2 e^{\imath\phi_2 + \imath\nu_2 t} \mid c\rangle\langle a \mid \\ = &-\frac{\hbar}{2}\begin{pmatrix} 0 & 0 & \Omega_1 e^{\imath\phi_1 + \imath\nu_1 t} \\ 0 & 0 & \Omega_2 e^{\imath\phi_2 + \imath\nu_2 t} \\ \Omega_1 e^{-\imath\phi_1 - \imath\nu_1 t} & \Omega_2 e^{-\imath\phi_2 - \imath\nu_2 t} & 0 \end{pmatrix} \end{aligned}$$

and

$$H_0 = \hbar\omega_b \mid b\rangle\langle b \mid +\hbar\omega_c \mid c\rangle\langle c \mid +\hbar\omega_a \mid a\rangle\langle a \mid = \hbar\begin{pmatrix} \omega_b & 0 & 0 \\ 0 & \omega_c & 0 \\ 0 & 0 & \omega_a \end{pmatrix}$$

Setting the ground state to 0 (i.e. $-\hbar\omega_b$ from the above terms) yields:

$$H_0 = \hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega_{cb} & 0 \\ 0 & 0 & \omega_{ab} \end{pmatrix}$$

where $\Omega_1, \Omega_2, \phi_1, \phi_2$ are real, known, dimensionless functions of time and ν_1, ν_2 are real, known constants in units of Hz.

By defining $\mid \psi\rangle$ to be

$$\mid \psi\rangle = \begin{pmatrix} \rho_b \\ \rho_c \\ \rho_a \end{pmatrix}$$

the density matrix ρ, which describes the evolution of the quantum system, is defined as

$$\begin{aligned} \rho &= \mid \psi\rangle\langle\psi^* \mid \\ &= \begin{pmatrix} \rho_b \\ \rho_c \\ \rho_a \end{pmatrix} \begin{pmatrix} \rho_b^* & \rho_c^* & \rho_a^* \end{pmatrix} \\ &= \begin{pmatrix} \rho_{bb} & \rho_{bc} & \rho_{ba} \\ \rho_{cb} & \rho_{cc} & \rho_{ca} \\ \rho_{ab} & \rho_{ac} & \rho_{aa} \end{pmatrix} \end{aligned} \tag{3}$$

Now, the statistical nature of the density matrix places the constraint

$$\rho_{aa} + \rho_{bb} + \rho_{cc} = 1.$$

Also, we notice that ρ is a Hermitian matrix (i.e. $\rho_{xy} = conj(\rho_{yx})$ for $x, y = a, b, c$).

Now, by putting equations (2) and (3) into (1) with an $R = 0$, we obtain the following five first-order nonlinear ordinary differential equations that are necessary to solve for the quantum state:

$$\dot{\rho}_{bb} = 0.5\imath\Omega_1 e^{\imath\phi_1}\tilde{\rho}_{ab} - 0.5\imath\Omega_1 e^{-\imath\phi_1}\tilde{\rho}_{ba} + \Gamma_{ab}\rho_{aa} + \Gamma_{cb}\rho_{cc} \tag{4}$$

$$\dot{\rho}_{cc} = 0.5\imath\Omega_2 e^{\imath\phi_2}\tilde{\rho}_{ac} - 0.5\imath\Omega_2 e^{-\imath\phi_2}\tilde{\rho}_{ca} + \Gamma_{ac}\rho_{aa} - \Gamma_{cb}\rho_{cc} \tag{5}$$

$$\rho_{aa} = 1 - (\rho_{bb} + \rho_{cc}) \tag{6}$$

$$\dot{\tilde{\rho}}_{cb} = 0.5\imath\Omega_2 e^{\imath\phi_2}\tilde{\rho}_{ab} - 0.5\imath\Omega_1 e^{-\imath\phi_1}\tilde{\rho}_{ca} - [\imath(\omega_{cb} - \nu_{12}) + \gamma_{cb}]\tilde{\rho}_{cb} \tag{7}$$

$$\dot{\tilde{\rho}}_{ab} = 0.5\imath\Omega_2 e^{-\imath\phi_2}\tilde{\rho}_{cb} - 0.5\imath\Omega_1 e^{-\imath\phi_1}(\rho_{aa} - \rho_{bb}) - [\imath(\omega_{ab} - \nu_1) + \gamma_{ab}]\tilde{\rho}_{ab} \tag{8}$$

$$\dot{\tilde{\rho}}_{ac} = 0.5\imath\Omega_1 e^{-\imath\phi_1}\tilde{\rho}_{bc} - 0.5\imath\Omega_2 e^{-\imath\phi_2}(\rho_{aa} - \rho_{cc}) - [\imath(\omega_{ac} - \nu_2) - \gamma_{ac}]\tilde{\rho}_{ac} \tag{9}$$

where the Γ and γ terms take into account population (diagonal density matrix terms) and coherence (off-diagonal terms) decay rates respectively and the $(\omega - \nu)$ terms account for angular frequency detuning between the two laser pulses.

2.2. **Numerical Approach.** We started implementing explicit numerical methods to solve simple cases for equations (4) through (9).

We first implemented an Euler method to solve the original system. That is, we discretized the time interval (with a step size of h) and found the solutions at $t_{n+1} = t_n + h$ by using the solutions and derivatives at time t.

$$\vec{\rho}_{n+1} = \vec{\rho}_n + h\dot{\vec{\rho}}_n \tag{10}$$

$$t_{n+1} = t_n + h \tag{11}$$

Next, we implemented a fourth order Runge-Kutta method with the same idea. So,

$$\vec{\rho}_n + 1 = \vec{\rho}_n + \frac{1}{6}h(k_1 + 2k_2 + 3k_3 + k_4) \tag{12}$$

$$t_{n+1} = t_n + h \tag{13}$$

where

$$\begin{aligned} k_1 &= f(t_n, \rho_n) \\ k_2 &= f(t_n + \frac{1}{2}h, \rho_n + \frac{1}{2}hk_1) \\ k_3 &= f(t_n + \frac{1}{2}h, \rho_n + \frac{1}{2}hk_2) \\ k_4 &= f(t_n + h, \rho_n + hk_3) \end{aligned}$$

Lastly, Matlab also has built-in numerical solvers. ode45 was chosen in addition to the other two numerical methods.

2.3. **Analytical Approach.** After working with some numerical methods on the simple cases, we started exploring the idea of an analytical solution to the system of equations. This way, we would be able to compare the accuracy of the numerical methods.

The first step in investigating the system analytically was to convert the nonlinear system into a linear one. To do this, recall that ρ is a Hermitian matrix. So, we made the

following substitutions into equations (4) through (9) in order to eliminate variables:

$$\rho_{xy} = Re(\rho_{xy}) + \imath Im(\rho_{xy})$$

$$\rho_{yx} = Re(\rho_{xy}) - \imath Im(\rho_{xy}) \quad \text{for } x, y = a, b, c$$

We then grouped the real and complex terms in each ordinary differential equation and separated each ODE into its real and complex parts, yielding the following linear system of eight ordinary differential equations in eight unknowns:

$$\dot{\rho}_{bb} = -\Omega_1 \sin(\phi_1) Re(\tilde{\rho}_{ab}) - \Omega_1 \cos(\phi_1) Im(\tilde{\rho}_{ab}) + \Gamma_{ab}\rho_{aa} + \Gamma_{cb}\rho_{cc} \tag{14}$$

$$\dot{\rho}_{cc} = -\Omega_2 \sin(\phi_2) Re(\tilde{\rho}_{ac}) - \Omega_2 \cos(\phi_2) Im(\tilde{\rho}_{ac}) + \Gamma_{ac}\rho_{aa} - \Gamma_{cb}\rho_{cc} \tag{15}$$

$$\begin{aligned} Re(\dot{\tilde{\rho}}_{cb}) = &-\frac{\Omega_1}{2}\sin(\phi_1) Re(\tilde{\rho}_{ac}) - \frac{\Omega_1}{2}\cos(\phi_1) Im(\tilde{\rho}_{ac}) - \frac{\Omega_2}{2}\sin(\phi_2) Re(\tilde{\rho}_{ab}) \\ &- \frac{\Omega_2}{2}\cos(\phi_2) Im(\tilde{\rho}_{ab}) - \gamma_{cb} Re(\tilde{\rho}_{cb}) + (\omega_{cb} - \nu_{12}) Im(\tilde{\rho}_{cb}) \end{aligned} \tag{16}$$

$$\begin{aligned} Im(\dot{\tilde{\rho}}_{cb}) = &-\frac{\Omega_1}{2}\cos(\phi_1) Re(\tilde{\rho}_{ac}) + \frac{\Omega_1}{2}\sin(\phi_1) Im(\tilde{\rho}_{ac}) + \frac{\Omega_2}{2}\cos(\phi_2) Re(\tilde{\rho}_{ab}) \\ &- \frac{\Omega_2}{2}\sin(\phi_2) Im(\tilde{\rho}_{ab}) - (\omega_{cb} - \nu_{12}) Re(\tilde{\rho}_{cb}) - \gamma_{cb} Im(\tilde{\rho}_{cb})) \end{aligned} \tag{17}$$

$$\begin{aligned} Re(\dot{\tilde{\rho}}_{ab}) = &-\frac{\Omega_1}{2}\sin(\phi_1)\rho_{aa} + \frac{\Omega_1}{2}\sin(\phi_1)\rho_{bb} + \frac{\Omega_2}{2}\sin(\phi_2) Re(\tilde{\rho}_{cb}) \\ &- \frac{\Omega_2}{2}\cos(\phi_2) Im(\tilde{\rho}_{cb}) - \gamma_{ab} Re(\tilde{\rho}_{ab}) + (\omega_{ab} - \nu_1) Im(\tilde{\rho}_{ab}) \end{aligned} \tag{18}$$

$$\begin{aligned} Im(\dot{\tilde{\rho}}_{ab}) = &-\frac{\Omega_1}{2}\cos(\phi_1)\rho_{aa} + \frac{\Omega_1}{2}\cos(\phi_1)\rho_{bb} + \frac{\Omega_2}{2}\cos(\phi_2) Re(\tilde{\rho}_{cb}) \\ &+ \frac{\Omega_2}{2}\sin(\phi_2) Im(\tilde{\rho}_{cb}) - (\omega_{ab} - \nu_1) Re(\tilde{\rho}_{ab} - \gamma_{ab} Im(\tilde{\rho}_{ab}) \end{aligned} \tag{19}$$

$$\begin{aligned} Re(\dot{\tilde{\rho}}_{ac}) = &\frac{\Omega_1}{2}\sin(\phi_1) Re(\tilde{\rho}_{cb}) + \frac{\Omega_1}{2}\cos(\phi_1) Im(\tilde{\rho}_{cb}) - \frac{\Omega_2}{2}\sin(\phi_2)\rho_{aa} \\ &+ \frac{\Omega_2}{2}\sin(\phi_2)\rho_{cc} + \gamma_{ac} Re(\tilde{\rho}_{ac}) + (\omega_{ac} - \nu_2) Im(\tilde{\rho}_{ac}) \end{aligned} \tag{20}$$

$$\begin{aligned} Im(\dot{\tilde{\rho}}_{ac}) = &\frac{\Omega_1}{2}\cos(\phi_1) Re(\tilde{\rho}_{cb}) - \frac{\Omega_1}{2}\sin(\phi_1) Im(\tilde{\rho}_{cb}) - \frac{\Omega_2}{2}\cos(\phi_2)\rho_{aa} \\ &+ \frac{\Omega_2}{2}\cos(\phi_2)\rho_{cc} - (\omega_{ac} - \nu_2) Re(\tilde{\rho}_{ac}) + \gamma_{ab} Im(\tilde{\rho}_{ac}) \end{aligned} \tag{21}$$

Lastly, we can rearrange this system of equations into a linear nonhomogeneous system of the form

$$\dot{\vec{\rho}} = A\vec{\rho} + \vec{b} \tag{22}$$

where

$$\vec{\rho} = \begin{pmatrix} \rho_{bb} \\ \rho_{cc} \\ Re(\tilde{\rho}_{cb}) \\ Im(\tilde{\rho}_{cb}) \\ Re(\tilde{\rho}_{ab}) \\ Im(\tilde{\rho}_{ab}) \\ Re(\tilde{\rho}_{ac}) \\ Im(\tilde{\rho}_{ac}) \end{pmatrix} \quad \vec{b} = \begin{pmatrix} \Gamma_{ab} \\ \Gamma_{ac} \\ 0 \\ 0 \\ \frac{-\Omega_1}{2}\sin(\phi_1) \\ \frac{-\Omega_1}{2}\cos(\phi_1) \\ \frac{-\Omega_2}{2}\sin(\phi_2) \\ \frac{-\Omega_2}{2}\cos(\phi_2) \end{pmatrix}$$

$$A = \begin{pmatrix} -\Gamma_{ab} & -\Gamma_{ab}+\Gamma_{cb} & 0 & 0 & -\Omega_1\sin(\phi_1) & -\Omega_1\cos(\phi_1) & 0 & 0 \\ -\Gamma_{ac} & -\Gamma_{ac}-\Gamma_{cb} & 0 & 0 & 0 & 0 & -\Omega_2\sin(\phi_2) & -\Omega_2\cos(\phi_2) \\ 0 & 0 & -\gamma_{cb} & \omega_{cb}-\nu_{12} & -\frac{\Omega_2}{2}\sin(\phi_2) & -\frac{\Omega_2}{2}\cos(\phi_2) & -\frac{\Omega_1}{2}\sin(\phi_1) & -\frac{\Omega_1}{2}\cos(\phi_1) \\ 0 & 0 & -\omega_{cb}+\nu_{12} & -\gamma_{cb} & \frac{\Omega_2}{2}\cos(\phi_2) & -\frac{\Omega_2}{2}\sin(\phi_2) & -\frac{\Omega_1}{2}\cos(\phi_1) & \frac{\Omega_1}{2}\sin(\phi_1) \\ \Omega_1\sin(\phi_1) & \frac{\Omega_1}{2}\sin(\phi_1) & \frac{\Omega_2}{2}\sin(\phi_2) & -\frac{\Omega_2}{2}\cos(\phi_2) & -\gamma_{ab} & \omega_{ab}-\nu_1 & 0 & 0 \\ \Omega_1\cos(\phi_1) & \frac{\Omega_1}{2}\cos(\phi_1) & \frac{\Omega_2}{2}\cos(\phi_2) & \frac{\Omega_2}{2}\sin(\phi_2) & -\omega_{ab}+\nu_1 & -\gamma_{ab} & 0 & 0 \\ \frac{\Omega_2}{2}\sin(\phi_2) & \Omega_2\sin(\phi_2) & \frac{\Omega_1}{2}\sin(\phi_1) & \frac{\Omega_1}{2}\cos(\phi_1) & 0 & 0 & \gamma_{ac} & \omega_{ac}-\nu_2 \\ \frac{\Omega_2}{2}\cos(\phi_2) & \Omega_2\cos(\phi_2) & \frac{\Omega_1}{2}\cos(\phi_1) & -\frac{\Omega_1}{2}\sin(\phi_1) & 0 & 0 & -\omega_{ac}+\nu_2 & \gamma_{ac} \end{pmatrix}$$

Now that we have a linear system of the form of equation (22), whenever A is a constant matrix, we have an analytical solution of the form

$$\vec{\rho}(t) = e^{(t-t_0)A}\vec{x}_0 + \int_{t_0}^{t} e^{(t-s)}\vec{b}(s)ds \tag{23}$$

The second term in equation (23) accounts for the nonhomogeneity present in (22).

3. Results and Discussion

The first two special cases share the following conditions:

$$\Gamma = \gamma = \omega_{ab} - \nu_1 = \omega_{ac} - \nu_2 = \omega_{cb} - \nu_{12} = 0$$

$$\Omega_1 = \Omega_2 \equiv \pi$$

$$\phi_1 \equiv k \ (constant) \qquad \phi_2 \equiv 0$$

With these conditions, our system becomes

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & -\pi\sin(k) & -\pi\cos(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\pi \\ 0 & 0 & 0 & 0 & 0 & -\frac{\pi}{2} & -\frac{\pi}{2}\sin(k) & -\frac{\pi}{2}\cos(k) \\ 0 & 0 & 0 & 0 & \frac{\pi}{2} & 0 & -\frac{\pi}{2}\cos(k) & \frac{\pi}{2}\sin(k) \\ \pi\sin(k) & \frac{\pi}{2}\sin(k) & 0 & -\frac{\pi}{2} & 0 & 0 & 0 & 0 \\ \pi\cos(k) & \frac{\pi}{2}\cos(k) & \frac{\pi}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\pi}{2}\sin(k) & \frac{\pi}{2}\cos(k) & 0 & 0 & 0 & 0 \\ \frac{\pi}{2} & \pi & \frac{\pi}{2}\cos(k) & -\frac{\pi}{2}\sin(k) & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{-\pi}{2}\sin(k) \\ \frac{-\pi}{2}\cos(k) \\ 0 \\ \frac{-\pi}{2} \end{pmatrix}$$

For the first special case,

$$\rho(t=0) = \frac{1}{2}\begin{pmatrix} 1 & -e^{ik} & 0 \\ -e^{-ik} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Theoretically, the density matrix should not change at all as time elapses. Figure 2 shows us that our methods follow this prediction.

For the second special case,

$$\rho(t=0) = \frac{1}{2}\begin{pmatrix} 1 & e^{ik} & 0 \\ e^{-ik} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We received predictions for the density matrix at $t = 1, 2, 3, 4$, which are marked on Figure 3 along with our methods simulating the quantum system.

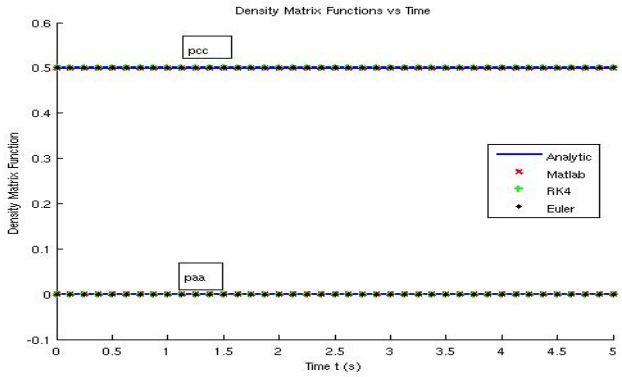

FIGURE 2. Special Case 1

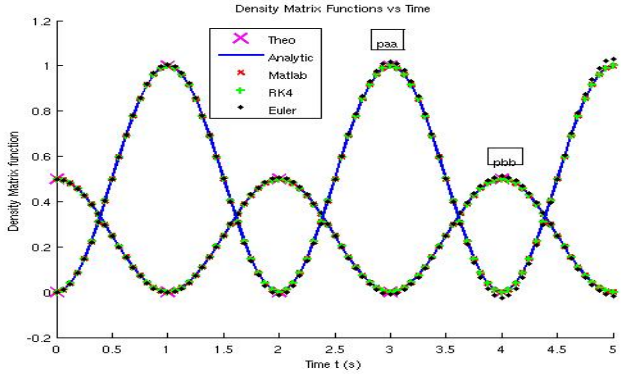

FIGURE 3. Special Case 2

We used the analytical solution to compare the error in the other numerical methods as time progressed (keeping in mind this analytical solution involved a numerical integration), which is seen in Figure 4.

The last case has the following conditions:

$$\Gamma = \gamma = \omega_{ab} - \nu_1 = \omega_{ac} - \nu_2 = \omega_{cb} - \nu_{12} = 0$$

$$\Omega_1 = \Omega_2 = 2\pi\Omega_R \text{sech}[\beta(t - t_0)]e^{i\mu \ln(\text{sech}[\beta(t-t_0)])}$$

where $\Omega_R = .55MHz, \mu = 1.93,$ and $\beta = 1.47$. In this case, ϕ_1, ϕ_2 take on the role of a pair of pi-pulses (with the center of the pulses beginning at $t_0 = 2.2\mu s$), where each pair pulse is shot every 4.4 μs with the system beginning at the ground state ($\mid c\rangle$). There are

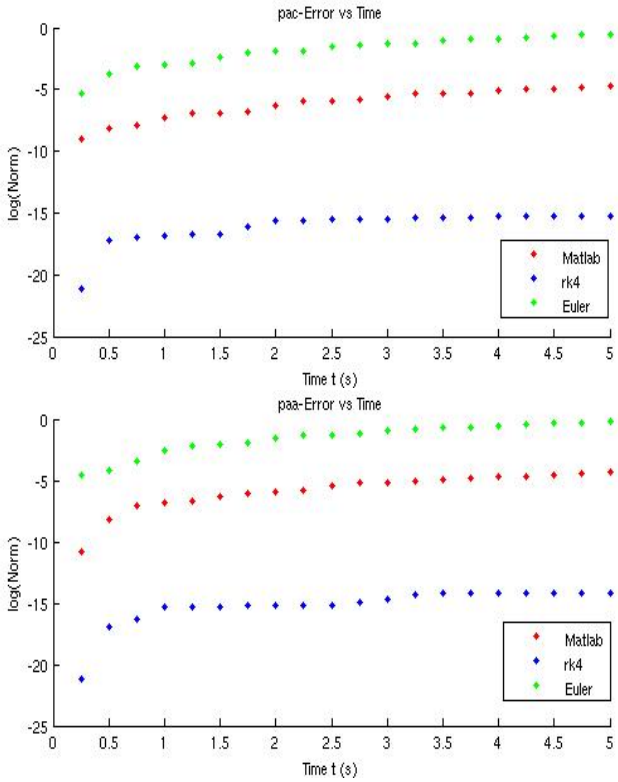

FIGURE 4. Accumulation of error as time progresses

four pair-pulses total, so the total run-time is 17.6 μs. Comparing our data to that from the Physics Department (Figure 5), we see that they both behave similarly.

4. Conclusion and Future Work

For the first two special cases, each method produces a simulation that is consistent with the predictions of the system for those specific parameters. From the error graphs, we see that the Euler method has a less accurate performance than ode45 and RK4 (which was expected). RK4 seems to perform better than ode45, but until we have a basis of comparison that doesn't depend on numerical integration, we can say that RK4 compares better than ode45 in relation to the analytical solution that requires numerical integration.

In the last case, the numerical methods provide us with the expected simulations, but the analytical solution did not. This may be because the current method of finding the analytical solution may be based on having a constant coefficient matrix in the system, which is not true in the last case.

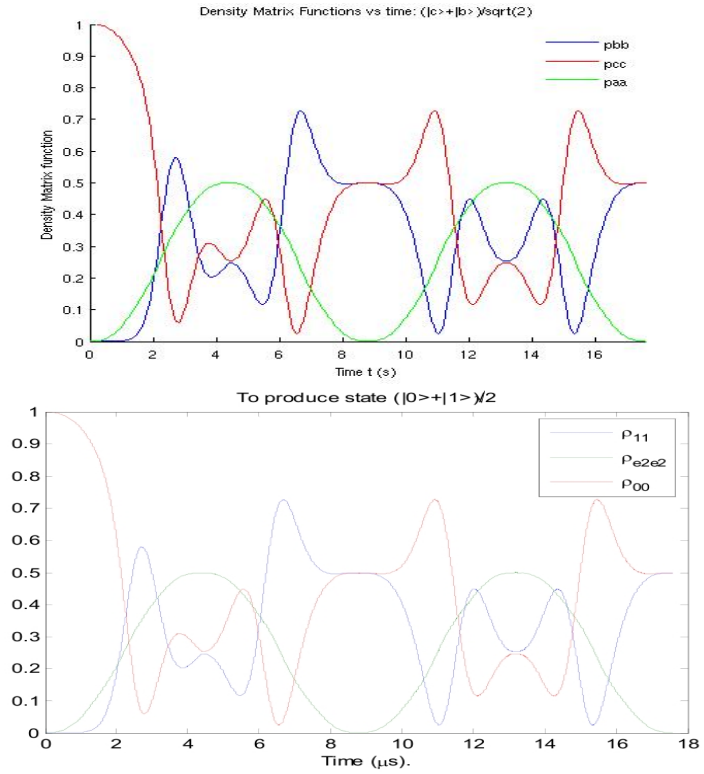

FIGURE 5. Comparison of Case 3 data

Some future goals would be to:

- do a more detailed error analysis on case 2 using a solution that does not require numerical integration.
- find the exact solution (with and/or without numerical integration) to case 3. It may be a good idea to generalize the method of finding the solution (for instance, using a method like variation of parameters).
- understand and account for key stability/accuracy issues to maximize accuracy but minimize computational cost.
- implement an implicit numerical scheme.
- continue to compare results with the Physics Department.

5. Acknowledgments

- National Science Foundation
- Department of Defense
- Dr. Maria Emelianenko
- Dr. Ming Tian
- Mr. Mahmoud Lababidi

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