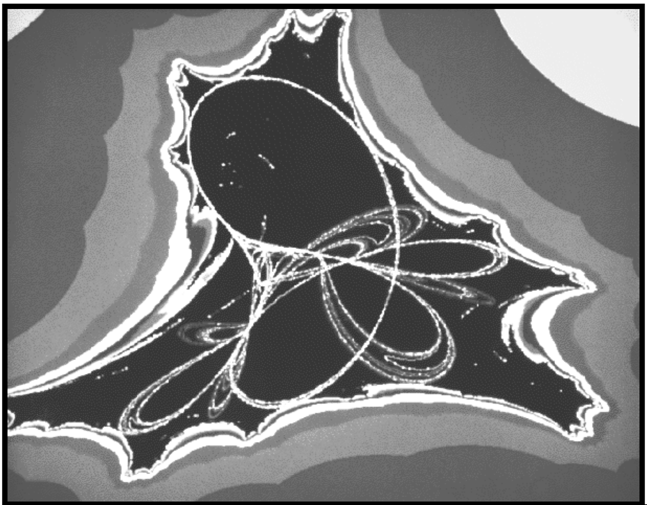
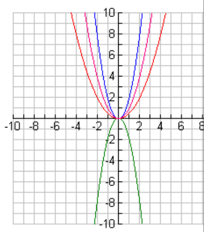
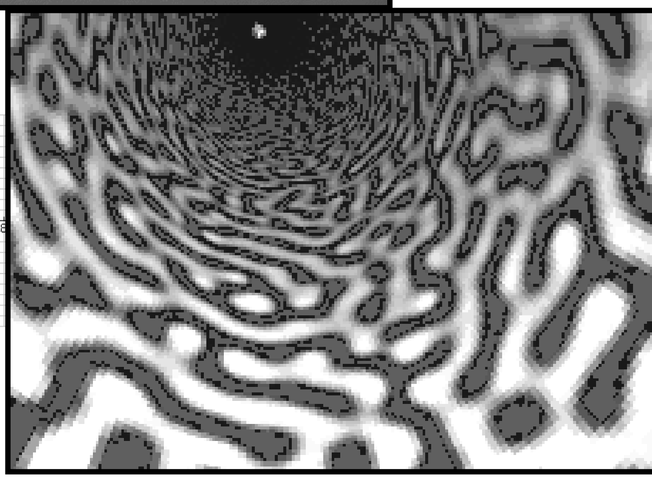






Algebra I, Algebra II, and Transcendentals Review

George Mason University

Revised September 2017

Contents

Courses that Require the Placement Test

The Math Placement Test is given by the Department of Mathematical Sciences to determine the readiness for the following courses:

- Math 104 Trigonometry and Transcendental Functions and Math 105 Pre-Calculus
- Math 108 Introductory Calculus with Business Applications
- Math 113 Analytic Geometry and Calculus I
- Math 123 Calculus Algebra/Trigonometry A
- Math 125 Discrete Mathematics
- CS 112 Computer Science I

Students should talk to an academic advisor to determine which Math course(s) they are required to take. All students admitted to the university are advised to take the Math Placement Test during the orientation process.

Policies

For current policies and required minimum scores please visit the webpage:

http://math.gmu.edu/placement_test.php

Basic Algebra

Basic Operations

Example: John's car insurance premium is $1515.00. He has the option of making four equal payments but will be charged an additional $25 processing fee. If he chooses the four-payment option, what is the amount of each payment?

Solution: The processing fee is added to the premium

$$\begin{array}{r} \$1515.00 \\ +\quad 25.00 \\ \hline \$\ 1540.00 \end{array}$$

This amount is then divided by 4

$$4\overline{)1540}\quad = 385$$

Each payment will be $385.00

Exercise 1: A shipment of 1344 CD's is to be packed into cartons containing 24 CD's each. How many cartons are necessary to ship these CD's?

Exercise 2: A loan of $10,020.00 is to be paid off in 60 equal monthly payments. How much is each payment (excluding interest)?

Exercise 3: Susan's Cell phone provider charges her $35.00 per month for the first 300 minutes of calls and $0.70 for each minute over 300. If she uses 340 minutes of calls in a given month, how much will she be charged?

Fractions

A fraction or rational number is of the form $\frac{p}{q}$ where *p* and *q* are integers, and

$q \neq 0$

p is called the numerator and *q* is called the denominator. The fraction is said to be reduced to lowest terms if *p* and *q* have no factors (other than 1) in common. The same number can be expressed as a fraction in many ways. These fractions are called equivalent fractions: $\frac{1}{4}$, $\frac{3}{12}$, $\frac{5}{20}$, $\frac{10}{40}$, $\frac{12}{48}$

Only the first fraction is in reduced form.

► To change from a mixed number to an improper fraction, multiply the whole number by the denominator of the fraction and add the result to the numerator of the fraction. This gives the numerator of the improper fraction.

Example: $1\frac{2}{3} = \frac{5}{3} \quad (5 = (1 \times 3) + 2)$

$5\frac{3}{8} = \frac{43}{8} \quad (43 = (5 \times 8) + 3)$

► The product of two fractions $\frac{a}{b} \times \frac{c}{d}$ is $\frac{ac}{bd}$. That is, multiply numerators to get the numerator of the product and multiply denominators to get the denominator of the product.

Examples: $\frac{2}{11} \times \frac{4}{9} = \frac{8}{99}$

$6 \times \frac{2}{5} = \frac{6}{1} \times \frac{2}{5} = \frac{12}{5}$

► The quotient of two fraction $\frac{a}{b} \div \frac{c}{d}$ is equal to $\frac{a}{b} \times \frac{d}{c}$ *or* $\frac{ad}{cb}$

Example: 49 ounces of a solution is to be poured into test tubes with capacity $3\frac{1}{2}$ ounces. How many test tubes will be filled?

Solution: $49 \div 3\frac{1}{2} = 49 \div \frac{7}{2} = 49 \times \frac{2}{7} = 14$ 14 test tubes will be filled.

Exercise 4: Perform the indicated operation and give the result in reduced form.

a. $\frac{1}{3} \div \frac{1}{5}$ b. $\frac{3}{8} \div 6$ c. $1\frac{2}{5} \times 2\frac{3}{4}$ d. $6\frac{2}{3} \div \frac{5}{8}$

► The sum of two fractions with a common denominator is $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

If fractions do not have the same denominator then they must be changed to an equivalent form.

Example: Find the sum $1\frac{1}{3} + \frac{5}{8}$

Change each fraction to an equivalent form

$$1\frac{1}{3} = \frac{4}{3} \cdot \frac{8}{8} = \frac{32}{24}$$

$$\frac{5}{8} = \frac{5}{8} \cdot \frac{3}{3} = \frac{15}{24}$$

Then add the equivalent fractions to get $1\frac{1}{3} + \frac{5}{8} = \frac{32}{24} + \frac{15}{24} = \frac{47}{24}$ or $1\frac{23}{24}$

Exercise 5: Perform the indicated operation and give the result in reduced form

a. $6\frac{1}{4} + \frac{7}{8}$ b. $\frac{7}{10} + \frac{3}{4}$ c. $\frac{5}{8} - \frac{1}{6}$ d. $2\frac{1}{4} - \frac{7}{12}$

Decimal Notation

Rational numbers or fractions can also be expressed in decimal notation.

$7\frac{13}{100}$ can be written as 7.13

.125 is equivalent to $\frac{125}{1000}$ or $\frac{1}{8}$

► When adding or subtracting numbers in decimal notation, make sure to align the decimal point.

Example: Find the sum $23.1 + 3.78$

Solution:

$$\begin{array}{r} 23.1 \\ +\quad 3.78 \\ \hline 26.88 \end{array}$$

► When multiplying two numbers in decimal notation, first multiply the numbers disregarding the decimal point(s). Next put the decimal point in the product. The number of places to the right of the decimal point in the product should be the SUM of the places to the right of the decimal point in each of the factors.

Example: Find the product 6.35×2.1

Solution:

$$\begin{array}{r} 635 \\ \times \quad 21 \\ \hline 635 \\ 12700 \\ \hline 13335 \end{array}$$

The product should have three places to the right of the decimal point.
The result will be 13.335

Example: Carlos is getting paid $10.80 per hour for the first 40 hours he is working and one time and a half that wage for each extra hour. If he works 43.5 hours this week, how much will he earn?

Solution: Carlos is paid $\$10.80 \times 1.5 = \$16.20$ per hour of overtime
He will earn: $(40 \times \$10.80) + (3.5 \times \$16.20) = 488.70$

Percents

The symbol % is used to indicate the fraction $\frac{1}{100}$ or the ratio: 1 *to* 100

23% is equivalent to $\frac{23}{100}$ *or* .23

100% is equivalent to $\frac{100}{100}$ *or* 1.00

Exercise 6: Change the decimal to percent

a. .25 b. .085 c. 1.52

Exercise 7: Change the percent to a decimal

a. 2% b. $5\frac{1}{2}\%$ c. 0.3%

Example: A class contains 12 female and 8 male students. What percent of the class is female?

Solution: There are a total of 20 students in the class.

12 out of 20 or $\frac{12}{20}$ are female

$\frac{12}{20} = \frac{60}{100} = 60\%$ of the class is female

Exercise 8: Find the following

a. 30% of 75 b. 4% of 196 c. 120% of 300

Example: Suppose 60% of a class is female. Of these females, 20% are freshman. What percent of the class is female upper class students?

Solution: If 20% of the females are freshman, then 80% of females are upper class students. 80% of 60%= $.80 \times .60 = .48$

so 48% of the class are female upper class students

Example: Suppose we know that 40% of a class or 16 students are male. How many students are in the class?

Solution: We know 40% of (*the number*) is 16

That is $.40 \times (\textit{the number}) = 16$

So $(\textit{the number}) = \frac{16}{.40} = 40$ students

***Example*: Before graduating, Joe worked in a convenience store and earned $5.85/hr. He started a new job that is paying him $18.95/hr. What is the percent increase in his hourly rate?**

***Solution*:** $\%\ increase = \frac{actual\ increase}{original}$ **so** $\frac{18.95 - 5.85}{5.85}$ **or 224% after rounding.**

Ratio

The ratio of two numbers a and b is written a to b, a:b or $\frac{a}{b}$

We can think of this as a fraction in order to simplify it.
The ratio 10:5 is equivalent to 2:1.

The ratio 32.5 to 5 is equivalent to *65 to 10* since $\frac{32.5}{5} = \frac{32.5}{5} \cdot \frac{2}{2} = \frac{65}{10}$

► **To show two ratios are equal use the following:**

$\frac{A}{B} = \frac{C}{D}$ **if and only if** $AD = BC$

***Example*: Show that 12.5 : 5 is equivalent to the ratio 5:2**

***Solution*: Since** $12.5 \times 2 = 25 = 5 \times 5$, $\frac{12.5}{5} = \frac{5}{2}$ **i.e. 12.5: 5 is equivalent to the ratio 5:2.**

***Example*: How much water should be mixed with 7 cups of salt to make a 12% saline solution?**

***Solution*: We have the proportion** $\frac{7}{x} = \frac{12}{100}$ $\left(\frac{part}{total}\right)$

Multiply to get $12x = 700$

$$x = 58\frac{1}{3}$$

So we need $58\frac{1}{3} - 7 = 51\frac{1}{3}$ **cups of water.**

Exponents

►When working with algebraic expressions involving exponents, the following properties of exponents are used:

1. $x^a x^b = x^{a+b}$
2. $\frac{x^a}{x^b} = x^{a-b}$
3. $\left(x^a\right)^b = x^{ab}$
4. $(xy)^a = x^a y^a$
5. $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$
6. $x^0 = 1$ *for any* $x \neq 0$

Exercise 9: Simplify each of the following expressions using the rules of exponents

a. $\left(3x^4\right)^2$ b. $\dfrac{4a^{10}}{2ab^5}$ c. $\dfrac{16x^5y^2}{5x^2\left(2xy^2\right)^3}$

***Exercise 10:* Find the product** $5x^3y^7z^0\left(12xy^6z^2\right)$

Algebraic Expressions

Most of Algebra involves manipulating expressions that contain variables.

Example: Evaluate the expression using the indicated value of the variable

$$(3k-1)(k+5) \quad \text{for} \quad k=-2$$

Solution:

$(3(-2)-1)((-2)+5)$ *substitute -2 in the expression for k*

$(-6-1)(-2+5)$

$(-7)(3) =$ -21

Exercise 11: Evaluate the expression using the indicated value of the variable

a. $2x^2-6x+3 \quad \text{for} \quad x=-2$ b. $\dfrac{5+\frac{1}{x}}{4(x+1)} \quad \text{for} \quad x=4$

Exercise 12: Combine the terms below and simplify the result.

a. $-4(x+3)+8x-2$

b. $16a-2b+4(a+3b)$

c. $\dfrac{4x}{5}-\dfrac{3x+1}{10}$

Linear Equations in One Variable

A linear equation in one variable can have a unique solution, no solution or infinitely many solutions. When it has no solution, the equation is called inconsistent.

Example: Solve the following equations by isolating the variable.

$$2x-15=-4x+21$$

Solution: $2x-15+4x=-4x+21+4x$ *Add 4x to both sides*

$6x-15=21$

$6x-15+15=21+15$ *Add 15 to both sides*

$$6x = 36$$

$$\left(\frac{1}{6}\right)6x = \left(\frac{1}{6}\right)36$$ *Multiply both sides by* $\frac{1}{6}$

$$x = 6$$ *The equation has a unique solution*

Example: Solve the equation $16x = 4(4x - 3)$

Solution:

$$16x = 16x - 12$$ *Distribute the 4*

$$16x - 16x = 16x - 12 - 16x$$ *Add -16x to both sides*

$$0 = -12$$ *The equation is* ***inconsistent****.*

Example: Solve the equation $10x + 3(x - 5) = 2(6x - 10) + x + 5$

Solution:

$$10x + 3x - 15 = 12x - 20 + x + 5$$ *Clear the parentheses by distributing*

$$13x - 15 = 13x - 15$$ *Combine like terms*

$$0 = 0$$

Since the last statement is always true, every real number will be a solution.

Linear Equations in Two Variables

A *solution* for an equation in two variables is a replacement for each of the variables so that when substituted, the resulting statement is true. We generally write the solution as an *ordered pair.* For example, $(2, 5)$ is a solution of the equation $3x + y = 11$. When the variable *x* is replaced by 2 in the equation and the variable *y* is replaced by 5, the resulting statement is true. The linear equation in two variables has an infinite number of solutions. The graph of these solutions is a line in the *x-y* plane.

Example: The formula in two variables gives the conversion between Fahrenheit and Celsius temperatures. $(x^\circ C \text{ to } y^\circ F)$

$$y = \frac{9}{5}x + 32$$

Find the temperature in degrees Celsius that corresponds to $212^\circ F$

Solution: $212 = \frac{9}{5}x + 32$ *Substitute the value 212 for y*

$180 = \frac{9}{5}x$ *Add -32 to both sides*

$100 = x$ *Multiply both sides by* $\frac{5}{9}$

A solution of the equation is $(100, 212)$

The Graph of a Line

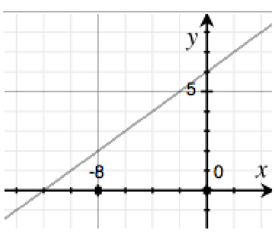
We can graph the solutions of any linear equation on a set of axes called the Euclidean Plane. In the case of the linear equation, the solutions form a line.\We can measure how much and in which direction the line tips using the slope.

The slope-intercept form of the equation is : $y = mx + b$

If the equation is in this form, the line has slope *m* and y-intercept (0, *b*)

The y-intercept is the point where the line intersects the y-axis.

Example: In the graph below, find the x and y intercepts and the slope.



Solution:
x intercept: (-12, 0)
y intercept: (0, 6)
Slope is 1/2

Example: Find the slope and y-intercept of the line with equation:

$$5x - 10y = 25$$

Solution: Write the equation in slope-intercept form: $y = \frac{1}{2}x - \frac{5}{2}$

The slope is $\frac{1}{2}$ and the y-intercept $\left(0, -\frac{5}{2}\right)$

Parallel lines have identical slopes. Lines are perpendicular if their slopes have the relationship $m_1 = -\frac{1}{m_2}$

Example: Find the slope of a line that is perpendicular to the line

$$5x - 10y = 25$$

Solution: The line $5x - 10y = 25$ has slope $\frac{1}{2}$.

Any line perpendicular to it will have slope m= - 2.

Algebra I

The Algebra I section of the test measures your ability to perform basic algebraic operations and to solve problems that involve algebraic concepts. Only those students attempting to place into Math 108, Math 112, Math 125, Math 113, CS 112 will be given questions from the Algebra I section. In addition to **all material in the Basic Algebra**, the following exercises and examples provide a refresher of the material that is tested in the Algebra I section.

Linear Inequalities in One Variable

Unlike linear equations, linear inequalities have an infinite number of solutions. We can use a number line or interval notation to indicate the solutions. Any real number can be added to or subtracted from both sides of the inequality. We can multiply or divide both sides by any nonzero real number but, if we multiply or divide by a negative number, the inequality sign must be reversed.

Example: Solve the inequality $3 \pounds \frac{-9}{5}x + 2$

Subtract 2 from both sides: $1 \pounds \frac{-9}{5}x$

Multiply both sides by -5/9: $\frac{-5}{9} \; 3 \; x$

The solution is the interval: $(-\infty, -\frac{5}{9}]$

Absolute Value Equations and Inequalities

The absolute value of a number is defined to be its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Example: Solve the equation for x $|15 - 3x| = 75$

Solution: $15 - 3x = 75$ or $15 - 3x = -75$

$x = -20$ or $x = 30$

Example: Solve the inequality $|2x + 5| \geq 15$

Solution: $2x + 5 \leq -15$ or $2x + 5 \geq 15$

$x \leq -10$ or $x \geq 5$

Distance Formula

► The distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The formula uses the Pythagorean Theorem

Exercise 1: Find the distance between the points $(-2, 5)$ and $(1, 8)$

Systems of Linear Equations

We sometimes group equations together and investigate whether they have a common solution. This is called a system of equations. The system of linear equations has a solution if there is an ordered pair (x, y) that satisfies each equation in the system. A system of linear equations can have exactly one solution, no solution or infinitely many solutions. There are several methods available to solve systems of linear equations. The next example illustrates the Elimination Method.

Example: Solve the system $\begin{cases} 3x + 2y = 48 \\ 9x - 8y = -24 \end{cases}$

Solution:

$3x + 2y = 48$ *multiply by -3:* $-9x - 6y = -144$

$9x - 8y = -24$ $\qquad\qquad\qquad\qquad \underline{9x - 8y = -24}$

Add: $-14y = -168$

Solve for y: $y = 12$

The value 12 is substituted for *y* in either of the original equations.

$3x + 2(12) = 48$ *Solve the resulting equation for x*

$3x + 24 = 48$

$3x = 24$

$x = 8$ *The solution to the system is* (8, 12)

Exercise 2: Solve the systems

a. $\begin{cases} 3x - y = 5 \\ -5x + 2y = -10 \end{cases}$

b. $\begin{cases} 3x - y = -2 \\ 6x - 2y = 10 \end{cases}$

Linear Inequalities in Two Variables

Linear inequalities in two variables can be solved by graphing. The solution is an infinite set of ordered pairs in the plane. We can indicate the solution by shading the region that contains these ordered pairs.

Example: Solve the inequality $5y \geq 3x - 15$

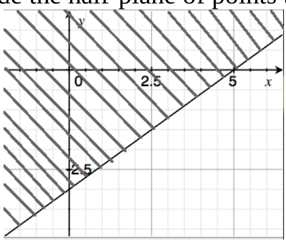
Solution: Graph the line that results from replacing the inequality sign with an equal sign. Here, graph the line $5y = 3x - 15$ This line partitions the plane into three sets:

The solutions of $5y > 3x - 15$ $5y < 3x - 15$ $5y = 3x - 15$

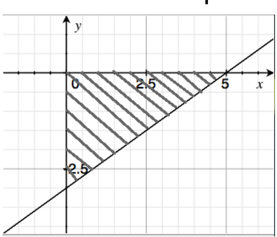
In the inequality, test an ordered pair from the plane that does not lie on the line.

Using (0,0), we get: $5(0) \geq 3(0) - 15$ which is a true statement.

Shade the half-plane of points that satisfy the inequality. See the graph below.



In a similar way we can solve a system of inequalities. The graph below shows the solutions to the system $\begin{cases} 5y \geq 3x - 15 \\ y \leq 0 \\ x \geq 0 \end{cases}$



Multiplying Algebraic Expressions

We use the Distributive Law to multiply algebraic expressions.

Example: Find the product $6x(2x-10)$

Solution: $6x(2x-10) = 6x(2x) + 6x(-10)$

$= 12x^2 - 60x$

To multiply 2 *binomial* factors, we apply the Distributive Law twice and combine like terms.

Example: Find the product $(x+7)(2x-1)$

Solution:

$$\begin{aligned}(x+7)(2x-1) &= x(2x-1)+7(2x-1)\\ &= x\cdot 2x - x\cdot 1 + 7\cdot 2x + 7\cdot(-1)\\ &= 2x^2 - x + 14x - 7\\ &= 2x^2 + 13x - 7\end{aligned}$$

The result of this multiplication is called a *quadratic expression*. We "undo" the multiplication in a process called *factoring.*

Factoring Algebraic Expressions

The next two examples show the factoring process. In the first, we factor out the highest common factor among the terms. In the second, we write the quadratic expression as a product of 2 binomial factors.

Example: Factor the expression $16x^2 + 24x$

Solution:

$$16x^2 + 24x = x(16x+24)$$ *The terms have the factor x in common*

$$= 8x(2x+3)$$ *They also have a factor of 8 in common*

Example: Factor the expression $t^2 + 18t - 40$

Solution:

$$t^2 + 18t - 40 = (t+20)(t-2)$$

Special Factorings:

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Rational Expressions

Rational expressions are fractions where the numerator and/or denominator involve variables. We can reduce, add, subtract, multiply and divide these rational expressions using the rules of fractions. You may want to review the rules of fractions in the Basic Algebra section on page 6.

Example: Reduce the rational expression $\dfrac{35-7x}{x^2-3x-10}$

Solution: In order to reduce the expression we must factor both the numerator and the denominator, then cancel any factors common to both.

$$\frac{35-7x}{x^2-3x-10}=\frac{7(5-x)}{(x+2)(x-5)}$$

$$=\frac{-7(x-5)}{(x+2)(x-5)} \qquad \textit{Factor the numerator}$$

$$=\frac{-7}{(x+2)} \qquad \textit{Cancel the common factor}$$

Example: Find the sum $\dfrac{4x+3}{x^2-9}-\dfrac{x+1}{x-3}$

Solution:

$$\frac{4x+3}{x^2-9}-\frac{x+1}{x-3}=\frac{4x+3}{(x+3)(x-3)}-\frac{(x+3)(x+1)}{(x+3)(x-3)} \qquad \textit{Use a common denominator}$$

$$=\frac{4x+3-(x^2+4x+3)}{(x+3)(x-3)}$$

$$=\frac{-x^2}{x^2-9}$$

*Example***:** Simplify the expression $\dfrac{5x^2}{\dfrac{1}{a+b}}$

Solution: Using the rules of dividing fractions we get:

$$5x^2\cdot\frac{a+b}{1}=5x^2(a+b)$$

More Exponents

►The definition of exponents is extended to include all rational numbers:

$$x^{-a}=\frac{1}{x^a} \qquad x^{\frac{a}{b}}=\sqrt[b]{x^a}$$

Exercise 3: If x= $\frac{2}{3}$ find the value of x^{-3}

Exercise 4: Find the value of $27^{\frac{2}{3}}\cdot 9^{\frac{1}{2}}$

Radicals

When simplifying algebraic expressions involving radicals, the following properties of radicals are used:

1. $\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{xy}$
2. $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$ *where* $y \neq 0$
3. $\sqrt[m]{x^n} = \left(\sqrt[m]{x}\right)^n$

Example: Find the product $\left(6+\sqrt{5}\right)\cdot\left(1-\sqrt{5}\right)$

Solution: $\left(6+\sqrt{5}\right)\cdot\left(1-\sqrt{5}\right) = 6\cdot1-6\cdot\sqrt{5}+1\cdot\sqrt{5}-\sqrt{5}\sqrt{5}$

$= 1-5\sqrt{5}$

Exercise 5: Find the sum: $\sqrt{28}+3\sqrt{7}$

Exercise 6: Simplify each of the following expressions

a. $\sqrt{8a^6b^5}$ b. $\sqrt{\frac{28u^7v^3}{7uv^2}}$ c. $\sqrt[3]{8x^5y^{15}}$

Exercise 7: Rationalize the denominator to simplify: $\frac{2\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

Quadratic Equations

An equation of the form $ax^2+bx+c=0$ is called a quadratic equation. The equation will have *two, one or no* real number solutions. There are several methods available to find the solutions of a quadratic equation. The following examples illustrate these methods.

Example: Solve the equation $2x^2 = 20$

Solution: $2x^2 = 20$

$x^2 = 10$ *Isolate the term involving the radical*

$x = \pm\sqrt{10}$ *Take the square root of both side of the equation*

Example: Solve the equation $x^2+4x+4=25$

Solution: $x^2+4x+4=25$

$x^2+4x-21=0$ *bring all terms to one side*

$(x+7)(x-3)=0$ *factor the left side*

$x+7=0$ *or* $x-3=0$ *set each factor equal to zero*

$x=-7$ or $x=3$ *the equation has 2 solutions*

Example: Solve the equation $9x^2+6x+1=0$

Solution: In the quadratic formula: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ we use

$a=9$, $b=6$ and $c=1$

$x=\dfrac{-6\pm\sqrt{6^2-4\cdot 9\cdot 1}}{2\cdot 9}=\dfrac{-6}{18}$ since $b^2-4ac=0$

we get only one solution $x=-\dfrac{1}{3}$ to the equation.

►The expression b^2-4ac is called the discriminant. It tells us how many solutions a quadratic equation will have.

If $b^2-4ac>0$ the equation has 2 real number solution

If $b^2-4ac=0$ the equation has 1 real number solution

If $b^2-4ac<0$ the equation has no real number solution

Exercise 8: Solve the equations

a. $4x^2-49=0$

b. $2x^2+9x-5=0$

c. $x^2+4x-6=0$

Function Notation

Functions can be defined using a list of ordered pairs, a graph, a table, or an equation.

The following equation defines a function: $f(x)=x^2-1$.

Each value of x is paired with the value $f(x)$.

$f(3)=3^2-1=8$ so the pair $(3,8)$ is in the function *f*.

$f(10)=10^2-1=99$ so the pair $(10,99)$ is in the function *f*.

$f(-10)=(-10)^2-1=99$ so the pair $(-10,99)$ is in the function *f*.

Example: For the function $f\left(x\right)=\dfrac{x^2-1}{x+3}$ ***find*** $f\left(-2\right)$ ***and*** $f\left(b+1\right)$

***Solution*:**

$$f\left(-2\right)=\frac{(-2)^2-1}{-2+3}=\frac{3}{1}=3$$

$$f\left(b+1\right)=\frac{(b+1)^2-1}{(b+1)+3}=\frac{b^2+2b}{b+4}$$

Complex Rational Expression

A complex rational expression is one in which a rational expression appears in the numerator and/or denominator. Simplifying the complex rational expression is equivalent to finding the quotient of the expression in the numerator with the expression in the denominator.

Example: Simplify the complex rational expression $\dfrac{\frac{1}{xy}}{\frac{1}{x}-\frac{1}{y}}$

Solution: Combine the terms in the denominator first.

$$\frac{1}{x}-\frac{1}{y}=\frac{y-x}{xy}$$

Next, find the quotient of the numerator with the denominator

$$\frac{1}{xy}\div\frac{y-x}{xy}=\frac{1}{xy}\cdot\frac{xy}{y-x}$$

$$=\frac{1}{y-x}$$

Equations with Rational Expressions

Given an equation involving rational expressions we can find an equivalent equation with no rational expressions. To find the equivalent equation we multiply the entire equation by the least common denominator (LCD). After solving the equivalent equation, we must check the solution in the original to see if it makes any denominator equal to zero. If so, we discard that solution.

Example: Solve the equation $\dfrac{1}{x}=\dfrac{1}{5}+\dfrac{3}{2x}$

Solution: The LCD of the rational expressions is *10x*

$$10x\cdot\frac{1}{x}=10x\cdot\frac{1}{5}+10x\cdot\frac{3}{2x}$$ *Multiply equation by 10x*

$$10 = 2x + 15 \qquad \textit{Solve the resulting equation}$$
$$-2x = 5$$
$$x = -\frac{5}{2}$$

Example: Solve the equation $\frac{4}{3x} - \frac{1}{3} = x$

Solution: The LCD of the rational expressions is *3x*

$$3x \cdot \frac{4}{3x} - 3x \cdot \frac{1}{3} = 3x \cdot x \quad \textit{Multiply equation by 3x}$$
$$4 - x = 3x^2 \quad \textit{Solve the resulting equation}$$
$$3x^2 + x - 4 = 0 \qquad \textit{Use the quadratic formula}$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 3 \cdot (-4)}}{2 \cdot 3}$$
$$x = \frac{-1 \pm \sqrt{49}}{2 \cdot 3}$$
$$x = 1 \ \textit{and} \ -\frac{4}{3}$$

Exercise 9: Solve the equation $\frac{2}{x+3} - \frac{2x+3}{x-1} = \frac{6x-5}{x^2+2x-3}$

Algebra II

The Algebra II section of the test measures your ability to solve problems that involve more advanced Algebra than the previous section. Only those students attempting to place into Math 113 and CS 112 will be given questions from the Algebra II section. The following exercises and examples provide a refresher of material that is tested in the Algebra II section.

More on Functions

A function is a set of ordered pairs where each first component is paired with exactly one second component. The set of first components is called the d*omain* and the set of second components is called the *range*.
Unless otherwise stated, the domain of the function $f(x) = x^2 - 1$, is all real numbers. Any real number *a* can be paired with another real number using the equation $f(x) = x^2 - 1$. The pair can be written $(a, a^2 - 1)$.

Suppose we have the function $g(t)=\frac{3}{t}$. We say $g(t)$ is *undefined* at $t=0$ since the expression $\frac{3}{0}$ is undefined. On the other hand, $g(t)$ is *defined* at any real number other than 0. The domain of $g(t)$ is all real numbers except $t=0$.

Suppose we have the function $f(x)=\sqrt{x+5}$. In order for the expression $\sqrt{x+5}$ to represent a real number, the expression $x+5$ must not be negative. For this function the domain is all real numbers $x \geq -5$.

►The average rate of change for a function *f(x)* is the change in *f(x)* over the change in x.

$$\textit{average rate of change} = \frac{\Delta f(x)}{\Delta x}$$

Alternatively, it is the slope of the secant line passing through the two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$

Exercise 1: Let $f(x)=\dfrac{1}{2x+6}$; Find each of the following:

a. the domain of $f(x)$ b. the range of *f(x)*

c. the average rate of change from x= -1 to x=5

Exercise 2: Let $g(x)=\dfrac{1}{\sqrt{3-x}}$; Find each of the following:

a. the domain of $g(x)$ b. the range of g*(x)*

c. the average rate of change from x= -2 to x=2

►We can **compose** two functions by applying one right after the other.

Example: For the functions *f* and *g* defined in the exercises 1 and 2, find

a. $f(g(-6))$ b. $g(f(-2))$

Solution: $f(g(-6))=f\left(\dfrac{1}{\sqrt{3-(-6)}}\right)=f\left(\dfrac{1}{3}\right)=\dfrac{3}{20}$

$$g(f(-2))=g\left(\frac{1}{2\cdot(-2)+6}\right)=g\left(\frac{1}{2}\right)=\frac{1}{\sqrt{5/2}} \text{ or } \frac{\sqrt{2}}{\sqrt{5}}$$

Example: The function values of *f* and *g* are given in the table below. Use them to find a. $f(g(0))$ b. $g(f(3))$

x	***f(x)***	***g(x)***
0	-3	3
1	0	2
2	-2	1
3	1	0

Solution: a. $f\left(g(0)\right) = f(3) = 1$

b. $g\left(f(3)\right) = g(1) = 2$

►Some functions have an **inverse**. The inverse of a function will interchange the real numbers in the ordered pair. If (a,b) is in the function *f* then (b,a) is in f^{-1}, the inverse of f. Keep in mind, that not all functions have inverses.

The functions $f(x) = 3x + 5$ and $f^{-1}(x) = \dfrac{x-5}{3}$ are inverses.

Notice *f* contains the pair (3,14) and f^{-1} contains the pair (14, 3). This is true for all ordered pairs of the function *f.*

Graphs of Functions

We can graph the set of ordered pairs of a function on a set of axes just as we do with linear equations in 2 variables.

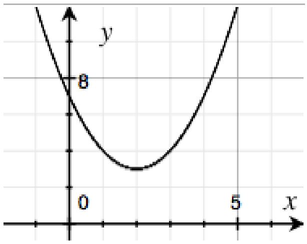
The graph of a quadratic function $f(x) = ax^2 + bx + c$ is a parabola.

The vertex of the parabola is located at $\left(\dfrac{-b}{2a}, f(\dfrac{-b}{2a})\right)$

This point on the graph is either a minimum or a maximum for the function. The leading coefficient *a*, determines which. For $a<0$, the function takes on a maximum at $x = \dfrac{-b}{2a}$ and the range of the function is $\left\{ y \,\middle|\, y \le f\left(\dfrac{-b}{2a}\right) \right\}$.

For $a>0$, the function takes on a minimum at $x = \dfrac{-b}{2a}$ and the range of the function will be $\left\{ y \,\middle|\, y \ge f\left(\dfrac{-b}{2a}\right) \right\}$.

Example: In the graph of $f(x) = x^2 - 4x + 7$ below, find the x and y intercepts and the vertex.



Solution:
y intercept: (0, 7)
x intercept: none
vertex: (2, 3)

note: (2, 3) is the *minimum* of f(x)

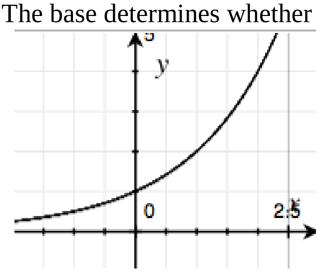
► **To graph the function** $y = c \cdot f(x-a)+b$ **we shift the graph of *f(x)***
right *a* units (for a>0), left |*a*| units for a<0)
up *b* units for b>0, down |*b*| units for b<0
vertically stretch the graph of *f(x)* by a factor of *c*, c>1
vertically shrink the graph of *f(x)* by a factor of *c*, for 0<*c*<1

Example: **The graph above can be viewed as a shift of** $y = x^2$
(2 units right and 3 units up). The function can be written in the form:
$f(x) = (x-2)^2 + 3$

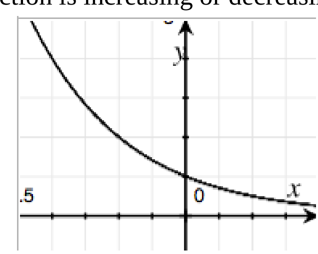
Transcendentals

The test measures your ability to work with exponential, logarithmic and trigonometric functions of the right triangle and the unit circle. The graphs of these functions and some common trigonometric formulas are given in the next few pages.

Exponential Functions
An exponential function is one in which the variable occurs in the exponent. The base determines whether the function is increasing or decreasing.



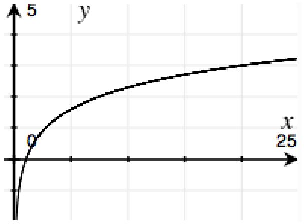
$f(x) = a^x$ *for* $a > 1$
Domain: all real numbers
Range: y>0



$f(x) = a^x$ *for* $a < 1$
Domain: all real numbers
Range: y>0

Logarithmic Functions

A logarithmic function is the inverse of an exponential function.



This is the natural log function, (the inverse of $y=e^x$)

f(x)= ln x
Domain: x>0
Range: all real numbers

Example: Evaluate: a. $\log_2 8$ b. $\log_5 1$ c. $\log_{10} .001$

Solution: a. $\log_2 8 = 3$ since $2^3 = 8$ b. $\log_5 1 = 0$ since $5^0 = 1$

c. $\log_{10} .001 = -3$ since $10^{-3} = .001$

Example: Solve the equation for *x*: $e^{x+3} = 10$

Solution: Isolate *x* by taking the log of both sides. Here use *ln* the natural log.

$$\ln e^{x+3} = \ln 10$$
$$x + 3 = \ln 10$$
$$x = -3 + \ln 10 \approx -.6974$$

Example: Solve the equation for *x*: $\log_4 (x + 13) = 6$

Solution: Use the inverse function to isolate *x*

$$4^{\log_4 (x+13)} = 4^6$$
$$x + 13 = 4^6$$
$$x = 4083$$

Angle Measure

An angle can be measured in degrees or in radians. A full revolution is 360° or 2π radians. We use this information to convert from one measure to the other. The following proportion shows the relationship between *x degrees* and *y radians.*

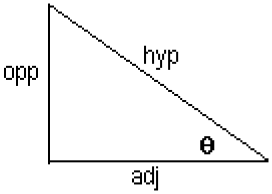
$$\frac{x}{y} = \frac{360}{2\pi}$$

Example: Find the degree measure of an angle whose radian measure is $\frac{2\pi}{3}$

Solution:

$$\frac{x}{(2\pi/3)} = \frac{360}{2\pi}$$
$$2\pi x = 360(2\pi/3) \text{ so } x = 120$$

Right Triangle Definitions

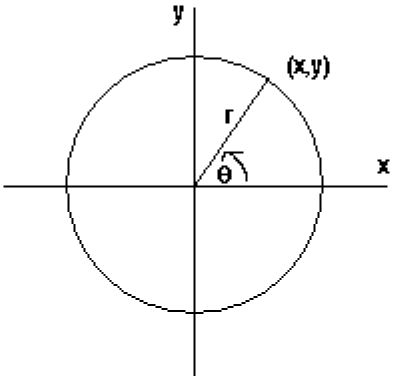


$$\sin\theta = \frac{opp}{hyp} \qquad \csc\theta = \frac{hyp}{opp}$$

$$\cos\theta = \frac{adj}{hyp} \qquad \sec\theta = \frac{hyp}{adj}$$

$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

Definitions of Circular Functions



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Special Angles

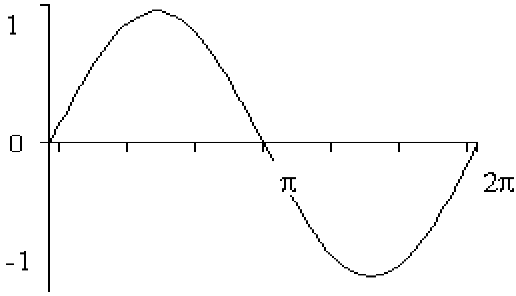
Degree	**Radian**	**sin θ**	**cos θ**	**tan θ**	**csc θ**	**sec θ**	**cot θ**
0	0	0	1	0	--	1	--
30	π/6	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$	$\sqrt{3}$
45	π/4	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\sqrt{2}$	$\sqrt{2}$	1
60	π/3	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$2\sqrt{3}/3$	2	$\sqrt{3}/3$
90	π/2	1	0	--	1	--	0

Sum and Difference Formulas

$$\sin(u \pm v) = \sin(u)\cos(v) \pm \cos(u)\sin(v)$$

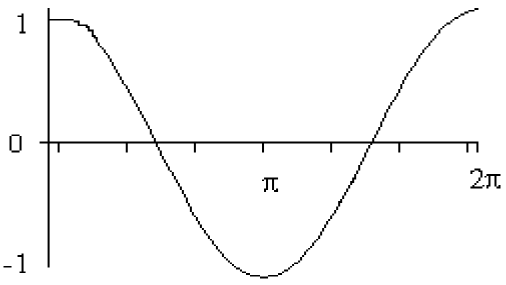
$$\cos(u \pm v) = \cos(u)\cos(v) \mp \sin(u)\sin(v)$$

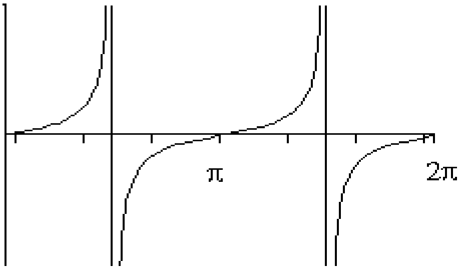
Graphs of the Trigonometric Functions



f(x)=sin(x) period: 2π

f(x)=cos(x) period: 2π





f(x)=tan(x) period: π

Exercise 1: Find the period of

a. $\sin(2x)$ b. $\cos\left(\frac{x}{4}\right)$ c. $\tan\left(\frac{x}{3}\right)$

Exercise 2: Which has the shortest period?

a. $y = \cos x$ b. $y = \frac{1}{2}\cos x$ c. $y = \cos(2x)$ d. $y = \cos\frac{1}{2}x$

Inverse Trigonometric Functions

The inverse sine function, written $\sin^{-1}(x)$ or $\arcsin(x)$, has domain [-1,1] and range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. It is defined to be the number in the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ whose sin is *x*.

In a similar way, $\cos^{-1}(x)$ or $\arccos(x)$ is defined to be the number in the interval $[0, \pi]$ whose cosine is *x*. The inverse tangent function, written $\tan^{-1}(x)$ or $\arctan(x)$, is defined to be the number is the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is *x*.

Exercise 3: Find each of the following:

a. $\arctan\left(\frac{\sqrt{3}}{3}\right)$ b. $\arccos(\cos\left(\frac{\pi}{3}\right))$ c. $\arcsin(0)$

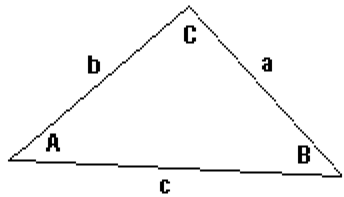
Double Angle Formulas

$$\sin(2u) = 2\sin(u)\cos(u)$$

$$\cos(2u) = \cos^2(u) - \sin^2(u) = 1 - 2\sin^2(u) = 2\cos^2(u) - 1$$

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



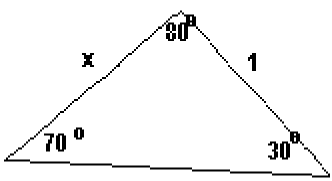
Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

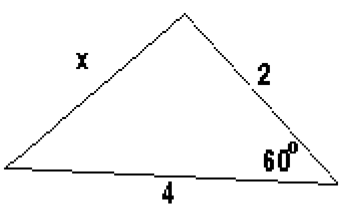
Pythagorean Identity

$$\sin^2 x + \cos^2 x = 1$$

Exercise 4: In the triangle below, find the length of the side marked with *x*.



Exercise 5: In the triangle below, find the length of the side marked with *x*.



Exercise 6: If $\sin A = \dfrac{5}{13}$ and $\tan A = \dfrac{5}{12}$, find $\cos A$

Answers to Exercises

Basic Algebra

1. 56 2. 167 3. $63.00

4. (a) $1\frac{2}{3}$ (b) $\frac{1}{16}$ (c) $3\frac{17}{20}$ (d) $10\frac{2}{3}$

5. (a) $7\frac{1}{8}$ (b) $1\frac{9}{20}$ (c) $\frac{11}{24}$ (d) $1\frac{2}{3}$

6. (a) 25 % (b) $8\frac{1}{2}$% (c) 152 %

7. (a) .02 (b) .055 (c) .003 8. (a) 22.5 (b) 7.84 (c) 360

9. (a) $9x^8$ (b) $\frac{2a^9}{b^5}$ (c) $\frac{2}{5y^4}$

10. $60x^4y^{13}z^2$

11. (a) 23 (b) 21/80

12. (a) $4x-14$ (b) $20a+10b$ (c) $\frac{5x-1}{10}$

Algebra I

1. $\sqrt{18}$ or $3\sqrt{2}$

2. (a) (0, -5) (b) inconsistent system

3. $\frac{27}{8}$ 4. 3^3 or 27 5. $5\sqrt{7}$

6. (a) $2a^3b^2\sqrt{2b}$ (b) $2u^3\sqrt{v}$ (c) $2xy^5\sqrt[3]{x^2}$

7. $-4+2\sqrt{6}$

8. (a) $\frac{-7}{2}, \frac{7}{2}$ (b) $-5, \frac{1}{2}$ (c) $-2-\sqrt{10}, -2+\sqrt{10}$

9. $-\frac{1}{2}, -6$

Algebra II

1. (a) all real numbers except $x=-3$ (b) all real numbers except y=0
   (c) -1/32

2. (a) all real numbers $x<3$ (b) all real numbers y>0
   (c) $\frac{1-\frac{1}{\sqrt{5}}}{4} = \frac{1}{4}(1-\frac{1}{\sqrt{5}})$

Transcendentals

1. (a) π (b) 8π (c) 3π
2. c
3. (a) 30° (b) $\dfrac{\pi}{3}$ (c) 0
4. $\dfrac{1}{2\sin 70^\circ}$ 5. $\sqrt{12}$ 6. $\dfrac{12}{13}$