Mean Minkowski Measures for Convex Sets and Grünbaum's Conjecture for Affine Diameters

Gabor Toth
Department of Mathematics
Rutgers University, Camden, NJ 08102

For a convex body $\mathcal{C}$ in $\mathbf{R}^n$, we let $\mathfrak{m}^*_{\mathcal{C}}$ denote its Minkowski measure. It can be defined as $\mathfrak{m}^*_{\mathcal{C}} = \inf_{\mathcal{O} \in \mathrm{int}\, \mathcal{C}} \mathfrak{m}_{\mathcal{C}}(\mathcal{O})$, where $\mathfrak{m}_{\mathcal{C}}(\mathcal{O})$ is the maximum chord ratio of $\mathcal{C}$ for all chords passing through the interior point $\mathcal{O}$; or dually, the maximum distance ratio of $\mathcal{O}$ within all pairs of parallel supporting hyperplanes of $\mathcal{C}$. We have $\mathfrak{m}_{\mathcal{C}}(\mathcal{O}) = \mathfrak{m}_{\mathcal{C}^{\mathcal{O}}}(\mathcal{O})$, where $\mathcal{C}^{\mathcal{O}}$ is the (polar) dual of $\mathcal{C}$ with respect to $\mathcal{O}$. The Minkowski measure is a measure of symmetry in the sense that

$$1 \leq \mathfrak{m}^*_{\mathcal{C}} \leq n,$$

where the lower bound is attained iff $\mathcal{C}$ is symmetric, and the upper bound is attained iff $\mathcal{C}$ is a simplex.
In this talk we define two sequences $\{\sigma_{\mathcal{C},k}(\mathcal{O})\}_{k\geq 1}$ and $\{\sigma^o_{\mathcal{C},k}(\mathcal{O})\}_{k\geq 1}$, $\mathcal{O} \in \mathrm{int}\, \mathcal{C}$. The kth members are "mean Minkowski measures in dimension k" which are dual in the sense that $\sigma^o_{\mathcal{C},k}(\mathcal{O}) = \sigma_{\mathcal{C}^{\mathcal{O}},k}(\mathcal{O})$. We also have

$$1 \leq \sigma_{\mathcal{C},k}(\mathcal{O}), \sigma^o_{\mathcal{C},k}(\mathcal{O}) \leq \frac{k+1}{2}.$$

The lower bound for $\sigma_{\mathcal{C},k}$ is attained iff $\mathcal{C}$ has a k-dimensional simplicial slice; and, for $\sigma^o_{\mathcal{C},k}$, iff $\mathcal{C}$ has a simplicial projection to a k-dimensional affine subspace. For $k \geq 2$, the upper bound is attained (in either case) iff $\mathcal{C}$ is symmetric with respect to $\mathcal{O}$.

An affine diameter of $\mathcal{C}$ is a chord with parallel supporting hyperplanes at the endpoints. In 1953 Klee showed that the condition $\mathfrak{m}^*_{\mathcal{C}} > n-1$ implies that there are $n+1$ affinely independent affine diameters meeting at a(ny) critical point $\mathcal{O}^* \in C$, where the infimum defining $\mathfrak{m}^*_{\mathcal{C}}$ is attained. In 1963 Grünbaum conjectured the existence of such point in any convex body. While this conjecture remains open (and difficult), as a byproduct of the properties of the dual mean Minkowski measures, we show that

$$\frac{n}{\mathfrak{m}^*_{\mathcal{C}} + 1} \leq \sigma^o_{\mathcal{C},n-1}(\mathcal{O}^*)$$

for any critical point $\mathcal{O}^*$ of $\mathcal{C}$, and if sharp inequality holds then the Grünbaum conjecture holds. This assumption is much weaker than Klee's.