Topology Preliminary Exam
January 19th, 2017

Problem 1. Prove that if $f : X \to Y$ is a bijective continuous map, then if X is compact and Y is Hausdorff, then f is a homeomorphism. Give a counterexample when X is only locally compact.

Problem 2. Prove that if X is locally compact Hausdorff, then there exists a set J and an embedding

$$i : X \hookrightarrow [0, 1]^J .$$

Problem 3. Show that the product topology on the set

$$X := \prod_{i=0}^{\infty} \mathbb{Z}$$

(where $\mathbb{Z}$ is the set of integers, with the discrete topology), is equivalent to the topology generated by the metric

$$d : X \times X \to [0, \infty),$$

defined the following:
for $\sigma = (\sigma_1, \sigma_2, \ldots)$ and $\tau = (\tau_1, \tau_2, \ldots)$ in X then $d(\sigma, \tau) = 0$ if and only if $\sigma = \tau$ and otherwise

$$d(\sigma, \tau) = min\{n \in N : \sigma(n) \neq \tau(n)\}.$$

Problem 4. Let X and Y be arbitrary topological spaces. Consider the canonical projection

$$pr_X : X \times Y \to X.$$

Prove that pr_X is a closed map, or give a counterexample.

Problem 5. a) Prove that a topological space X is Hausdorff if and only if its diagonal

$$\Delta_X := \{(x, y) \in X \times X \mid x = y\}$$

is closed in $X \times X$.

b) Let X be Hausdorff and $f, g : Y \to X$ continuous maps. Show that the subspace

$$\{y \in Y \mid f(y) = g(y)\} \subseteq Y$$

is closed.