Topology Preliminary Exam
August 24th, 2016

Problem 1. a) Prove that a topological space X is Hausdorff if and only if its diagonal

$$\Delta_X := \{(x, y) \in X \times X \mid x = y\}$$

is closed in $X \times X$.

b) Let X be Hausdorff and $f, g : Y \to X$ continuous maps. Show that the subspace

$$\{y \in Y \mid f(y) = g(y)\} \subseteq Y$$

is closed.

Problem 2. Prove that every connected open subset of a locally path connected space is path connected.

Problem 3. Prove that if X is locally compact Hausdorff, then there exists a set J and an embedding

$$i : X \hookrightarrow [0,1]^J .$$

Problem 4. Let S be a topological space with exactly two points. Prove that one of the following three scenarios must be true for all connected topological spaces X:

a) There are precisely two continuous functions $f : X \to S$.

b) The set of continuous functions $f : X \to S$ is in natural bijection with the power set $P(X)$ of X.

c) The set of continuous functions $f : X \to S$ is in natural bijection with all the open subsets of X.

Problem 5. Define what it means for a topological space X to be a one-point compactification of a locally compact Hausdorff space Y. Prove that in this case, the space $X - Y$ is compact Hausdorff.

Problem 6. Let X be a set, and equip it with the cofinite topology (finite compliment topology). Prove that X is metrizable if and only if X is finite.