January 2018

Department of Mathematical Sciences
George Mason University

Ordinary Differential Equations - Preliminary Exam

Closed books, closed notes. Show work for full credit. No calculators are allowed.

1. Let $A \in \mathbb{R}^{n\times n}$ denote an arbitrary matrix, and suppose that $E \subset \mathbb{R}^n$ is a linear subspace of $\mathbb{R}^n$ which is invariant under A, i.e., for every vector $y \in E$ we have $Ay \in E$.
   (a) Show that if $x_0 \in E$, then the solution $x(t)$ of the linear differential equation $\dot{x} = Ax$ subject to the initial condition $x(0) = x_0$ stays in E for all $t \in \mathbb{R}$.
   (b) Illustrate the statement of (a) for the matrix
   $$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$
   Which subspaces of $\mathbb{R}^2$ are invariant, and how can you solve the linear system $\dot{x} = Ax$ in these subspaces?

2. Answer the following questions.
   (a) State the existence and uniqueness theorem for ordinary differential equations.
   (b) Give an example of a first order initial value problem that has more than one solution. Justify your answer.
   (c) Is there a continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(t) = (\cos t, \sin 2t)$ solves the differential equation $\dot{x} = f(x)$? Justify your answer.

3. Consider the autonomous system
   $$\begin{aligned} \dot{x} &= -y - \alpha^2 x y^2 \\ \dot{y} &= x^3 , \end{aligned}$$
   where α is a real parameter.
   (a) For which values of α is this system Hamiltonian? For each case, find the Hamiltonian and sketch the phase portrait.
   (b) For each value of α, find all equlibrium solutions of the above system. Can the principle of linearized stability be used to determine their stability?
   (c) Show that for all $\alpha \in \mathbb{R}$ the origin is a stable equilibrium. (Hint: Can you use the Hamiltonian function(s) from (a)?)

4. For the autonomous system in polar coordinates given by
   $$\begin{aligned} \dot{r} &= r - r^2 \\ \dot{\theta} &= \sin^2\theta \end{aligned}$$
   answer the following questions.
   (a) Find all equilibrium solutions of the system and determine their stability.
   (b) For every initial condition $x_0 \in \mathbb{R}^2$, describe the ω-limit set of the orbit starting at x_0.
   (c) Show that the unit circle centered at the origin is an attracting set. Is it an attractor?