Department of Mathematical Sciences
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January 2017

Ordinary Differential Equations - Preliminary Exam

Closed books, closed notes. Show work for full credit. No calculators are allowed.

1. For the system

$$\begin{aligned}\dot{x} &= x^2 - xy \\ \dot{y} &= x^2 - y\end{aligned}$$

answer the following questions.

(a) Find all equilibria of this system.

(b) Find the linearization $Df(x_0, y_0)$ for each equilibrium (x_0, y_0).

(c) Find the stability of the equilibrium at the origin. Describe the flow near the origin. Make sure you state the methods and theorems that you use.

2. Consider the linear system of equations $z' = Az$, where $z = (x, y)$ with

$$A = \begin{pmatrix} 0 & a \\ 1 & -1 \end{pmatrix}.$$

Classify the equilibrium point at the origin as a function of a. Sketch phase portraits for representative values of a. Produce one phase portrait for each fixed point type.

3. Consider the following system given in polar coordinates

$$\begin{aligned}\dot{r} &= -r^3 + r + r\sin(2\theta)/2 \\ \dot{\theta} &= 1 + \cos^2\theta.\end{aligned}$$

(a) Find all equilibria. Show that there are no invariant circles centered at the origin.

(b) Prove that the system has a periodic orbit. Make sure to state any theorems you use.

4. Consider the second order equation,

$$\ddot{x} + x - x^3 = 0.$$

Write this equation as a system of first order equations, letting $\dot{x} = y$. Then sketch the phase portrait and determine the ω-limit set of the points $(0, 0)$, $(0, 1/2)$, $(0, \frac{\sqrt{2}}{2})$ and $(0, 1)$.