Ordinary Differential Equations Preliminary Exam January 2020

(1) Consider the matrix

$$A = \begin{pmatrix} -3 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -5 & -2 \\ 0 & 0 & 0 & 2 & -5 \end{pmatrix}.$$

For the system $\dot{x} = Ax$, find the stable, center, and unstable subspaces. Describe the behavior inside each subspace, including any invariant lines and planes inside these spaces. Justify your answer.

(2) For the equation $\ddot{x} + x - \dot{x}\left(1 - x^2 - \frac{1}{2}\dot{x}^2\right) = 0$, noting that x is scalar here, do the following.

(a) Write the equation as a system in x and y by setting $\dot{x} = y$. Find all equilibria for the system.

(b) Show that the annulus $1/2 < \sqrt{x^2 + y^2} < 2$ is invariant.

(c) Show that the annulus $1/2 < \sqrt{x^2 + y^2} < 2$ contains a closed orbit. Justify your answer, stating any theorem that you use.

(3) Consider the autonomous system

$$\begin{aligned} \dot{x} &= -y - \alpha^2 x y^2 \\ \dot{y} &= x^3, \end{aligned}$$

where α is a real parameter.

(a) For which values of α is this system Hamiltonian? For each case, find the Hamiltonian and sketch the phase portrait.

(b) For each value of α, find all equlibrium solutions of the above system. Can the principle of linearized stability be used to determine their stability?

(c) Show that for all $\alpha \in \mathbb{R}$ the origin is a stable equilibrium. (Hint: Can you use the Hamiltonian function(s) from (a)?)

(4) Answer the following questions.

(a) State the existence and uniqueness theorem for ordinary differential equations.

(b) Give an example of a first order initial value problem that has more than one solution. Justify your answer.

(c) Is there a continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(t) = (\cos t, \sin 3t)$ solves the differential equation $\dot{x} = f(x)$? Justify your answer.