Ordinary Differential Equations Preliminary Exam August 2019

(1) Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} .$$

For the system $\dot{x} = Ax$, find the stable, center, and unstable subspaces. Describe the behavior inside each subspace, including any invariant lines and planes inside these spaces. Justify your answer.

(2) Consider the damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0 ,$$

where m, b, and k are all positive parameters.

(a) Write this equation as a system in x and y by setting $\dot{x} = y$

(b) Suppose that m and k are fixed and b is varied (while remaining positive). Classify the fixed point at the origin and sketch the corresponding phase portrait for each qualitatively distinct case of b.

(c) Assume again that the parameters m and k are fixed. In addition, suppose that $x(0) = 1$ and $\dot{x}(0) = 0$. For which values of b is x never negative? Among the set of b values where x is never negative, find the value of b such that (x, y) has the quickest rate of convergence to the equilibrium steady state solution $(x, y) = (0, 0)$.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous with $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$, and consider the nonlinear oscillator equation $\ddot{x} + f(x) = 0$ in the form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -f(x) \end{aligned}$$

Show that in this system, the equilibrium $(x_0, 0)$ is not attracting. Give examples in which the equilibrium is stable or unstable.

(4) Consider the following system given in polar coordinates

$$\begin{aligned} \dot{r} &= r(r^2 - 5r\cos\theta - 6) \\ \dot{\theta} &= 1 \end{aligned}$$

Prove that there is a periodic orbit for the system. State any theorems you use. (Hint: Find an annulus of the form $\{r : a < r < b\}$ for some values of a and b and show that this annulus contains a periodic orbit.)