Ordinary Differential Equations Preliminary Exam January 2019

(1) Let $(x(t), y(t))$ be a nontrivial solution of the initial value problem

$$\begin{array}{rcl} \dot{x} & = & 2x - 2y \quad , \quad x(0) = x_0 \\ \dot{y} & = & 2x - 3y \quad , \quad y(0) = y_0 \end{array}$$

i.e., assume $(x_0, y_0) \neq (0, 0)$. Find all possible values of the two limits

$$\lim_{t \to -\infty} \frac{y(t)}{x(t)} \qquad \text{and} \qquad \lim_{t \to +\infty} \frac{y(t)}{x(t)} .$$

Which initial conditions (x_0, y_0) lead to which limits?

(2) Suppose that A is an $n \times n$-matrix which satisfies $A^2 = I$.

(a) Find an explicit formula for e^{tA} in terms of A, but without any series involving A.

(b) Determine the stability of the origin of the linear system $\dot{x} = Ax$. Is the origin stable? Are there matrices A with $A^2 = I$ for which the origin is asymptotically stable? If so, determine all such matrices.

(3) Consider the system

$$\begin{array}{rcl} \dot{x} & = & xy \, , \\ \dot{y} & = & -y - x^2 \, . \end{array}$$

Verify that the equilibrium at the origin has a one-dimensional center subspace. Find an approximation for the local center manifold and the flow on the local center manifold. Determine the stability of the origin, making sure that you state all theorems that you use.

(4) Consider the planar system of ordinary differential equations given by

$$\begin{array}{rcl} \dot{x} & = & x(x + 2y - 3) \, , \\ \dot{y} & = & -y(2x + y - 3) \, . \end{array}$$

(a) Find all equilibrium points of this system and determine their stability.

(b) Show that the three lines $x = 0$, $y = 0$, and $x + y = 3$ are all invariant, as well as the triangle $D = \{(x, y) \, : \, x \geq 0 \, , \, y \geq 0 \, , \, x + y \leq 3\}$ and its boundary ∂D.

(c) For every solution which has its initial condition on the boundary ∂D of D, what are its α- and ω-limit sets?