Department of Mathematical Sciences
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Ordinary Differential Equations - Preliminary Exam

Closed books, closed notes. Show work for full credit. No calculators are allowed.

1. Suppose that A is an $n \times n$-matrix which satisfies $A^2 = -A$.
   (a) Find an explicit formula for e^{tA} in terms of A, but without any series involving A. (Hint: Show first that $A^k = (-1)^{k+1}A$ for all $k \geq 1$.)
   (b) Determine the stability of the origin of the linear system $\dot{x} = Ax$.

2. Consider the autonomous planar ordinary differential equation

$$\begin{aligned} \dot{x} &= y + \lambda x \, , \\ \dot{y} &= x - x^2 \, . \end{aligned}$$

   For the parameter values $\lambda = 0$, $\lambda = -1/2$, and $\lambda = 1/2$, perform the following tasks.
   (a) Find the equilibrium points of the planar system and determine their stability.
   (b) Sketch the phase portraits.

3. State precisely the (classical) Hartman-Grobman Theorem. Give an example of a differential equation that does not satisfy the hypothesis of the theorem and show why the result does not hold for that example.

4. Consider the Susceptible-Infected-Recovered (SIR) model given by

$$\begin{aligned} \dot{S} &= -\beta S I \, , \\ \dot{I} &= \beta S I - \gamma I \, , \\ \dot{R} &= \gamma I \, , \end{aligned}$$

   where $\beta > 0$ and $\gamma > 0$ are model parameters and the functions $S(t)$, $I(t)$, and $R(t)$ represent the proportion of the population that is susceptible, infected, and recovered, respectively, from a given disease.
   (a) Show that if $S(0) + I(0) + R(0) = 1$, then we have $S(t) + I(t) + R(t) = 1$ for all times t in the maximal existence interval of the solution.
   (b) Show that the set $D = \{(S, I, R) \in \mathbb{R}^3 : S \geq 0\, ,\ I \geq 0\, ,\ R \geq 0\, ,\ S + I + R \leq 1\}$ is positively invariant.
   (c) Consider the initial condition $S(0) = 1 - \epsilon$, $I(t) = \epsilon$, and $R(t) = 0$, where $\epsilon > 0$ is a very small number. Determine whether a disease outbreak occurs or whether the disease is quickly eradicated. The answer will depend on the parameters β and γ.